Interactive Graphics Using Parametric Equations (Day 2)

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Computer Science
Bezier Curves

Google “bezier curves"

- Casselman's Bezier curves
- Andysspline Bezier Curves
- Bezier Photo: Automotive Design Engineer
Interactive Graphics Curves

Uses:
-- Design of Fonts and other printer symbols
-- Consumer goods: shapes of cell phones, cars, etc.
Bezier cutting
Bezier cutting
Bezier cutting
Bezier cutting
Bezier Curves

Curve is specified by 2 equations:

\[ x = x_0 (1 - t)^3 + 3x_1 t (1 - t)^2 + 3x_2 t^2 (1 - t) + x_3 t^3 \]

\[ y = y_0 (1 - t)^3 + 3y_1 t (1 - t)^2 + 3y_2 t^2 (1 - t) + y_3 t^3 \]

\((x_0, y_0)\) \hspace{1cm} \((x_3, y_3)\)

\((x_1, y_1)\) \hspace{1cm} \((x_2, y_2)\)
Plot some points

\[ x = t^4 + 1, \quad y = t^3 + t \]

<table>
<thead>
<tr>
<th>( t )</th>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>17</td>
<td>-10</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>17</td>
<td>10</td>
</tr>
<tr>
<td>3</td>
<td>??</td>
<td>??</td>
</tr>
</tbody>
</table>
A Bezier Curve

\[ P_0( x_0, y_0 ) \]
\[ P_1( x_1, y_1 ) \]
\[ P_2( x_2, y_2 ) \]
\[ P_3( x_3, y_3 ) \]
Bezrier Curves

Mathematically, we verified that:

Slope of a handle is same as tangent at endpoint

i.e., the tangent at \((x_0, y_0)\) has same slope as the segment joining \((x_0, y_0)\) to \((x_1, y_1)\)
Graphics Curve in General Form

Curve is specified by 2 equations, one is:

\[ x = x_0 (1 - t)^3 + 3x_1 t(1 - t)^2 + 3x_2 t^2 (1 - t) + x_3 t^3 \]

Which can be re-written as

\[
\begin{align*}
x &= x_0 \ t^3 (1 - t)^3 \\
    &+ 3 \ x_1 \ t^1 (1 - t)^2 \\
    &+ 3 \ x_2 \ t^2 (1 - t)^1 \\
    &+ \ x_3 \ t^3 (1 - t)^0
\end{align*}
\]
General Form of Bezier Curve

This equation

\[ x = x_0 \cdot t^0 \cdot (1 - t)^3 + 3 \cdot x_1 \cdot t^1 \cdot (1 - t)^2 + 3 \cdot x_2 \cdot t^2 \cdot (1 - t)^1 + x_3 \cdot t^3 \cdot (1 - t)^0 \]

can be re-written as

\[ x = \frac{3!}{0!(3 - 0)!} \cdot x_0 \cdot t^0 \cdot (1 - t)^3 + \frac{3!}{1!(3 - 1)!} \cdot x_1 \cdot t^1 \cdot (1 - t)^2 + \frac{3!}{2!(3 - 2)!} \cdot x_2 \cdot t^2 \cdot (1 - t)^1 + \frac{3!}{3!(3 - 3)!} \cdot x_3 \cdot t^3 \cdot (1 - t)^0 \]
General Form of Bezier Curve

This form

\[
x = \frac{3!}{0!(3-0)!} x_0 \ t^0 \ (1-t)^3 \\
+ \frac{3!}{1!(3-1)!} x_1 \ t^1 \ (1-t)^2 \\
+ \frac{3!}{2!(3-2)!} x_2 \ t^2 \ (1-t)^1 \\
+ \frac{3!}{3!(3-3)!} x_3 \ t^3 \ (1-t)^0
\]

can be re-written as

\[
x = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} x_k \ t^k \ (1-t)^{(n-k)}
\]
General Form of Bezier Curve

Hence, the equations are (using \( n=3 \), for 4 points)

\[
x = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} x_k t^k (1-t)^{(n-k)}
\]

\[
y = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} y_k t^k (1-t)^{(n-k)}
\]

Generic notation,

\[
\overset{\circ}{P} = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \overset{\circ}{p}_k t^k (1-t)^{(n-k)}
\]

\[
\overset{\circ}{P}(t) = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \overset{\circ}{p}_k t^k (1-t)^{(n-k)}
\]
General Form of Bezier Curve

\[ P(t) = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} p_k t^k (1-t)^{n-k} \]

Can be re-written as,

\[ P(t) = \sum_{k=0}^{n} p_k \frac{n!}{k!(n-k)!} t^k (1-t)^{n-k} \]

Or,

\[ P(t) = \sum_{k=0}^{n} p_k \text{BEZ}_{k,n}(t) \]

where the organizer can choose \( n \), and the user
General Form of Bezier Curve

\[ P(t) = \sum_{k=0}^{n} p_k \ BEZ_{k,n}(t) \]

is,

\[ P(t) = \sum_{k=0}^{n} p_k \frac{n!}{k!(n-k)!} t^k (1-t)^{(n-k)} \]

Most common is \( n=3 \), some use \( n=2 \), and \( n=4 \).

Abandon notion of handles, and use notion of guidepoints.
General Form of Bezier Curve

Most common is $n=3$, some use $n=2$, and $n=4$. 
Bezier Curve in Matrix Notation

Consider the familiar case

\[ x = x_0 (1 - t)^3 + 3x_1 t(1 - t)^2 + 3x_2 t^2 (1 - t) + x_3 t^3 \]

It expands to

\[ x = x_0 - 3x_0 t + 3x_0 t^2 - x_0 t^3 \]
\[ + 3x_1 t - 6x_1 t^2 + 3x_1 t^3 \]
\[ + 3x_2 t^2 - 3x_2 t^3 \]
\[ + x_3 t^3 \]
Towards Matrix Notation

\[ x = x_0 - 3x_0 t + 3x_0 t^2 - x_0 t^3 + 3x_1 t - 6x_1 t^2 + 3x_1 t^3 + 3x_2 t^2 - 3x_2 t^3 + x_3 t^3 \]

Gives

\[ x = t^3 (-1x_0 + 3x_1 - 3x_2 + x_3) + t^2 (3x_0 - 6x_1 + 3x_2) + t (-3x_0 + 3x_1) + t^0 (x_0) \]
Towards Matrix Notation

\[ x = t^3 (-1x_0 + 3x_1 - 3x_2 + x_3) \]
\[ + t^2 (3x_0 - 6x_1 + 3x_2) \]
\[ + t (-3x_0 + 3x_1) \]
\[ + t^0 (x_0) \]

Can be written as

\[
\begin{bmatrix}
-1 & 3 & -3 & 1 \\
3 & -6 & 3 & 0 \\
-3 & 3 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x_0 \\
x_1 \\
x_2 \\
x_3
\end{bmatrix}
\]
Matrix notation

Suppose we have

\[3 \text{ Apples} + 5 \text{ Bananas} = 10\]
\[6 \text{ Apples} + 7 \text{ Bananas} = 15\]

What is cost of Apple? cost of Banana?

Can write as

\[3A + 5B = 10\]
\[6A + 7B = 15\]

In matrix notation, get

\[
\begin{bmatrix}
3 & 5 \\
6 & 7
\end{bmatrix}
\begin{bmatrix}
A \\
B
\end{bmatrix}
= 
\begin{bmatrix}
10 \\
15
\end{bmatrix}
\]
Matrix notation

Using matrix notation, then would “solve this Matrix system”
\[
\begin{bmatrix}
3 & 5 \\
6 & 7
\end{bmatrix}
\begin{bmatrix}
A \\
B
\end{bmatrix}
= 
\begin{bmatrix}
10 \\
15
\end{bmatrix}
\]

for
\[
\begin{bmatrix}
A \\
B
\end{bmatrix}
\]

by finding inverse matrix of
\[
\begin{bmatrix}
3 & 5 \\
6 & 7
\end{bmatrix}
\]

This is studied in class on Matrix Algebra or Linear Algebra.
Matrix Notation

Matrices are also very useful in Computer Graphics for dealing with rotations and 3-dimensional projections.

For now, we only care about the notation; i.e., that we can write in one form (English) or the other (matrix).

\[
\begin{align*}
3 \text{ Apples} + 5 \text{ Bananas} &= \$10 \\
6 \text{ Apples} + 7 \text{ Bananas} &= \$15
\end{align*}
\]

\[
\begin{bmatrix}
3 & 5 \\
6 & 7
\end{bmatrix}
\begin{bmatrix}
A \\
B
\end{bmatrix}
= 
\begin{bmatrix}
10 \\
15
\end{bmatrix}
\]
Back To Bezier Curves

We had

\[ x = t^3 (-1x_0 + 3x_1 - 3x_2 + x_3) \]
\[ + t^2 (3x_0 - 6x_1 + 3x_2) \]
\[ + t (-3x_0 + 3x_1) \]
\[ + t^0 (x_0) \]

Wrote as

\[ x = [t^3 \ t^2 \ t \ 1] \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} \]
Back To Bezier Curves

This

\[ x = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} \]

has same form for \( y \)

Generic equation:

\[ \bar{P}(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \cdot M_{Bez} \cdot \begin{bmatrix} \bar{p}_0 \\ \bar{p}_1 \\ \bar{p}_2 \\ \bar{p}_3 \end{bmatrix} \]
Recall the Polynomial Notation

For curve, \( P(t) = \sum_{k=0}^{n} p_k \text{BEZ}_{k,n}(t) \)

\[
P(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \cdot M_{\text{Bez}} \cdot \begin{bmatrix} \vec{p}_0 \\ \vec{p}_1 \\ \vec{p}_2 \\ \vec{p}_3 \end{bmatrix}
\]

\( P(t, s) = \sum_{j=0}^{m} \sum_{k=0}^{n} p_{j,k} \text{BEZ}_{j,m}(s) \text{BEZ}_{k,n}(t) \)
Bezizer Surfaces

We have \( \mathbf{P}(t, s) = \sum_{j=0}^{m} \sum_{k=0}^{n} p_{j,k} \mathbf{BEZ}_{j,m}(s) \mathbf{BEZ}_{k,n}(t) \)

Or, for \( m=n=3 \)

\( \mathbf{P}(t, s) = [t^3 \ t^2 \ t \ 1] \cdot \mathbf{M}_{\mathbf{Bez}} \cdot \begin{bmatrix}
\mathbf{p}_{00} & \mathbf{p}_{01} & \mathbf{p}_{02} & \mathbf{p}_{03} \\
\mathbf{p}_{10} & \mathbf{p}_{11} & \mathbf{p}_{12} & \mathbf{p}_{13} \\
\mathbf{p}_{20} & \mathbf{p}_{21} & \mathbf{p}_{22} & \mathbf{p}_{23} \\
\mathbf{p}_{30} & \mathbf{p}_{31} & \mathbf{p}_{32} & \mathbf{p}_{33}
\end{bmatrix} \cdot \mathbf{M}_{\mathbf{Bez}}^T \begin{bmatrix}
s^3 \\
\mathbf{s^2} \\
\mathbf{s} \\
\mathbf{1}
\end{bmatrix} \)
Bezier Surfaces

We need

Figure adapted from Princeton Web site
Some Properties of Bezier Surfaces

1) Four corners are like Tent anchors, i.e., they are tied down to fixed points. (Interpolation)

2) Along any border, the surface behaves as a single Bezier curve.

3) Just as with single curve, the surface fits in the Convex Hull of the specified points
Somne Properties of Bezier Surfaces

4) Because the four borders are explicit Bezier curves, they can be linked to neighboring patches, by making the common border the same Bezier curve, i.e., the same four control points.
Demo: Bezier Surface

Google Bezier surface demo
http://
Bézier Patch Interactive Demonstration
Verify that \((t=0, s=0)\) is tied down.

We have

\[
P(t, s) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \cdot \begin{bmatrix} p_{00} & p_{01} & p_{02} & p_{03} \\ p_{10} & p_{11} & p_{12} & p_{13} \\ p_{20} & p_{21} & p_{22} & p_{23} \\ p_{30} & p_{31} & p_{32} & p_{33} \end{bmatrix} \cdot \begin{bmatrix} s^3 \\ s^2 \\ s \\ 1 \end{bmatrix}
\]

i.e.,

\[
P(t, s) = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} p_{00} & p_{01} & p_{02} & p_{03} \\ p_{10} & p_{11} & p_{12} & p_{13} \\ p_{20} & p_{21} & p_{22} & p_{23} \\ p_{30} & p_{31} & p_{32} & p_{33} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}
\]
Verify

So have

\[ \overset{\circ}{P}(t, s) = [t^3 \ t^2 \ t \ 1] \cdot M_{Bez} \cdot \begin{bmatrix} \vec{p}_{00} & \vec{p}_{01} & \vec{p}_{02} & \vec{p}_{03} \\ \vec{p}_{10} & \vec{p}_{11} & \vec{p}_{12} & \vec{p}_{13} \\ \vec{p}_{20} & \vec{p}_{21} & \vec{p}_{22} & \vec{p}_{23} \\ \vec{p}_{30} & \vec{p}_{31} & \vec{p}_{32} & \vec{p}_{33} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \]
So have

\[
P(t, s) = [0 \ 0 \ 0 \ 1] \cdot M_{Bez} \cdot \begin{bmatrix} p_{00} \\ p_{10} \\ p_{20} \\ p_{30} \end{bmatrix}
\]
Teapot

1) [away3d] Utah teapot and Bezier Patches - New addition!!
2) Geepers - Utah Teapot BezierPatch Demo
3) Rendering Cubic Bezier Patches

So done verification!!
The End