Applying hyperbolic functions to quantum tunneling and electromagnetic wave problems in physics and engineering (Lecture 1)

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I. Warmup questions concerning calculus

II. Hyperbolic functions $\cosh(x)$ and $\sinh(x)$

III. Excellent properties of the derivative is $\cosh(x)$ and $\sinh(x)$

IV. How they help with radiation problems
1. If \(10^a = b\), then \(\log_{10}(b) = \)

A. \(\log_{10}(b) = 0.1a\)
B. \(\log_{10}(2b) = 2a\)
C. \(\log_{10}(b) = a\)
D. \(\log_{10}(b/100) = 100 + a\)
1. If $10^a = b$, then $\log_{10}(b) =$

A. $\log_{10}(b) = 0.1a$
B. $\log_{10}(2b) = 2a$
C. $\log_{10}(b) = a$
D. $\log_{10}(b/100) = 100 + a$
Let’s take it apart!

\[ e^{4w} = 80 \]
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\[ e^{4w} = 80 \quad \rightarrow \quad \ln[80] = 4w \]
Let’s take it apart!

\[ e^{4w} = 80 \quad \rightarrow \quad \ln [80] = 4w \]

\[ \ln 80 = 4.38203 \]
Let’s take it apart!

\[ e^{4w} = 80 \quad \Rightarrow \quad \ln(80) = 4w \]
\[ \ln 80 = 4.38203 \]
\[ \therefore 4w = 4.38203 \]
Let’s take it apart!

\[ e^{4w} = 80 \quad \Rightarrow \quad \ln[80] = 4w \]

\[ \ln 80 = 4.38203 \]

\[ \therefore 4w = 4.38203 \]

So, you need a \( w \) value of 1.0955!
Let’s pull this apart!
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\[ \sinh(-x) = \frac{e^{-x} - e^{-(x)}}{2} \]
Let’s pull this apart!

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\[
\sinh(-x) = \frac{e^{-x} - e^x}{2}
\]

\[
\sinh(-x) = \frac{-(e^{-x} + e^x)}{2} = -\frac{e^x - e^{-x}}{2}
\]
Let’s pull this apart!

\[ \sinh(-x) = \frac{e^{-x} - e^{-(x)}}{2} \]

\[ \sinh(-x) = \frac{e^{-x} - e^x}{2} \]

\[ \sinh(-x) = \frac{-(e^{-x} + e^x)}{2} = -\frac{e^x - e^{-x}}{2} \]

\[ \sinh(-x) = -\sinh(x) \quad \text{YAY!} \]
What it means geometrically

\[ \sinh(-x) = -\sinh(x) \]

Left of origin, in the negatives…

…it is up side down!
Differentiation of a sum $= \sum$ of derivatives

$$\frac{d}{dx} \left[ \frac{e^x + e^{-x}}{2} \right]$$
Differentiation of a sum = sum of derivatives

\[ \frac{d}{dx} \left[ \frac{e^x + e^{-x}}{2} \right] \quad \rightarrow \quad \frac{1}{2} \left[ \left( \frac{d}{dx} e^x \right) + \left( \frac{d}{dx} e^{-x} \right) \right] \]
Differentiation of a sum = sum of derivatives

\[
\frac{d}{dx} \left[ \frac{e^x + e^{-x}}{2} \right] \rightarrow \frac{1}{2} \left[ \left( \frac{d}{dx} e^x \right) + \left( \frac{d}{dx} e^{-x} \right) \right] = e^x
\]
Differentiation of a sum $= \text{sum of derivatives}$

$$\frac{d}{dx} \left[ \frac{e^x + e^{-x}}{2} \right] \rightarrow \frac{1}{2} \left[ \left( \frac{d}{dx} e^x \right) + \left( \frac{d}{dx} e^{-x} \right) \right]$$

$$e^x - e^{-x}$$
Differentiation of a sum = sum of derivatives

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\[
e^x - e^{-x}
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\frac{1}{2} \left( e^x - e^{-x} \right)
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\]

\[= \frac{1}{2} \left( e^x - e^{-x} \right) \]

YAY!
Why does this catch the eye of a mathematician or a scientist?
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A. Differentiate once, you get the other guy.
Why does this catch the eye of a mathematician or a scientist?

A. Differentiate once, you get the other guy.
B. Differentiate again, you get back home!
Why does this catch the eye of a mathematician or a scientist?

A. Differentiate once, you get the other guy.

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\[
\frac{d^2}{dx^2} \cosh(x) = \cosh(x)
\]
Why does this catch the eye of a mathematician or a scientist?

A. Differentiate once, you get the other guy.

B. Differentiate again, you get back home!

\[
\frac{d^2}{dx^2} \cosh(x) = \cosh(x)
\]

\[
\frac{d^2}{dx^2} \sinh(x) = \sinh(x)
\]
Why does this catch the eye of a mathematician or a scientist?

C. Very useful!!!
D. Because derivatives have various physical interpretations, like vectors!

\[
\frac{d^2}{dx^2} \cosh(x) = \cosh(x)
\]

\[
\frac{d^2}{dx^2} \sinh(x) = \sinh(x)
\]
What our goal is this afternoon

Penetrate material with electromagnetic waves that reshape into hyperbolic functions.
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Penetrate material with electromagnetic waves that reshape into hyperbolic functions.

Ch. 2 in Applications of Calculus II
What are we talking about?
What are we talking about?
What are we talking about?

cosh

sinh
What are we talking about?

weird…but handy in physics!

cosh

sinh
Recall: Euler’s formula...

\[ e^{iz} = \cos(z) + i \sin(z) \]
Recall: Euler’s formula…

\[ e^{iz} = \cos(z) + i\sin(z) \]

We need this today!
Recall: Euler’s formula...

\[ e^{iz} = \cos(z) + i \sin(z) \]

\[ \cos(z) = \frac{1}{2} \left( e^{iz} + e^{-iz} \right) \]
Recall: Euler’s formula...

\[ e^{iz} = \cos(z) + i \sin(z) \]

\[
\cos(z) = \frac{1}{2} \left( e^{iz} + e^{-iz} \right)
\]

\[
\sin(z) = \frac{1}{2} \left( e^{iz} - e^{-iz} \right)
\]
How are they proportional?

\[ \frac{d}{dx} e^{kx} = k \cdot e^{kx} \]
How are they proportional?

\[
\frac{d}{dx} e^{kx} = k e^{kx} \quad \Rightarrow \quad \frac{k e^{kx}}{e^{kx}} = k
\]
What we are talking about
What we are talking about

$\cos(kx), \sin(kx)$
What we are talking about

\( \cos(kx), \sin(kx) \)

\( \exp(kx), \exp(-kx) \)
What we are talking about

- $\cos(kx), \sin(kx)$
- $\cosh(kx), \sinh(kx)$
- $\exp(kx), \exp(-kx)$
What we are talking about

\[ k = 2.7 \text{ for each} \]

\[ \cos(kx), \sin(kx) \]

\[ \cosh(kx), \sinh(kx) \]

\[ \exp(kx), \exp(-kx) \]
What we are talking about

Each graphed over \([-1, 2]\)

\[ k = 2.7 \text{ for each} \]
Here we go. Let’s look at derivatives.
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\[ \frac{d}{dt} \left[ \cos(\omega t) \right] = -\omega \sin(\omega t) \]
Here we go. Let’s look at derivatives.

\[
\frac{d}{dt} \left[ \cos(\omega t) \right] = -\omega \sin(\omega t)
\]

\[
\frac{d^2}{dt^2} \left[ \cos(\omega t) \right] = -\omega^2 \cos(\omega t)
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\]

Let’s check sines, too!
Here we go. Let’s look at derivatives.
Here we go. Let’s look at derivatives.

\[ \frac{d}{dt} \left[ \sin(\omega t) \right] = \omega \cos(\omega t) \]
Here we go. Let’s look at derivatives.

\[
\frac{d}{dt} \left[ \sin(\omega t) \right] = \omega \cos(\omega t)
\]

\[
\frac{d^2}{dt^2} \left[ \sin(\omega t) \right] = -\omega^2 \sin(\omega t)
\]
Here we go. Let’s look at derivatives.
Here we go. Let’s look at derivatives.

\[ \frac{d^2}{dt^2} [\cos (\omega t)] = -\omega^2 \cos (\omega t) \]

\[ \frac{d^2}{dt^2} [\sin (\omega t)] = -\omega^2 \sin (\omega t) \]
Here we go. Let’s look at derivatives.

\[ \frac{d^2}{dt^2} [\cos(\omega t)] = -\omega^2 \cos(\omega t) \]

\[ \frac{d^2}{dt^2} [\sin(\omega t)] = -\omega^2 \sin(\omega t) \]

In general: after two diff’s you get back to same function… but with an extra factor of \(-\omega^2\).
General derivatives relationships here:

\[
\frac{d^2}{dt^2} \left[ f(\omega t) \right] = -\omega^2 \ f(\omega t)
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\]

\[
\frac{d^2}{dt^2} \left[f(\omega t)\right] + \omega^2 \ f(\omega t) = \ ?
\]
General derivatives relationships here:

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\frac{d^2}{dt^2} \left[ f(\omega t) \right] = -\omega^2 f(\omega t)
\]

\[
\frac{d^2}{dt^2} \left[ f(\omega t) \right] + \omega^2 f(\omega t) = ?
\]

What do these two add up to?
General derivatives relationships here:

$$\frac{d^2}{dt^2} \left[ f(\omega t) \right] = -\omega^2 f(\omega t)$$

$$\frac{d^2}{dt^2} \left[ f(\omega t) \right] + \omega^2 f(\omega t) = 0$$

What do these two add up to?
We like that because…
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A. That kind of derivative relationship is needed in spring systems, e.g.,
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✓ Oscillators
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- Oscillators
- Vibrating systems

Simple seismometer
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   ✓ Oscillators
   ✓ Vibrating systems

B. $F = -kx$ in springs

Simple seismometer
We like that because…

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   ✓ Vibrating systems

B. \( F = -kx \) in springs

C. \( F = ma \ldots \) as always

Simple seismometer
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   ✓ Oscillators
   ✓ Vibrating systems

B. $F = -kx$ in springs

C. $F = ma$...as always

D. $a = \text{second deriv. of } x!$

Simple seismometer
We like that because...

A. That kind of derivative relationship is needed in spring systems, e.g.,
   ✓ Oscillators
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B. $F = -kx$ in springs

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D. $a = \text{second deriv. of } x!$

\[
m \frac{d^2}{dt^2} (x[t]) = -kx(t)
\]

Simple seismometer
Brainiac iClicker question coming…
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\[ m \ a = -k \ x \]

\[ m \ \frac{d^2}{dt^2} \ (x[t]) = -k \ x(t) \]
For this derivatives relationship, what will the $\omega^2$ factor be?

\[ m \ a \quad = \quad -k \ x \]
\[ m \ \frac{d^2}{dt^2} \ (x[t]) \quad = \quad -k \ x(t) \]

A. It comes out to be $k/m$.

And the temporal scale in $\omega$ depends upon the mass ($m$) of the object on the spring and on the strength ($k$) of the spring.
Onward to radiation

\[ E'' = \frac{1}{c^2} \frac{d^2}{dt^2} E \]

in vacuum
Onward to radiation

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in vacuum

A. The relationship for radiation involves the electric field \((E)\) and magnetic field \((B)\)
Onward to radiation

\[ E'' = \frac{1}{c^2} \frac{d^2}{dt^2} E \]

in vacuum

A. The relationship for radiation involves the electric field \((E)\) and magnetic field \((B)\)

B. They are coupled
Onward to radiation

The relationship for radiation involves the electric field \( E \) and magnetic field \( B \).

They are coupled in space-time.

\[
E'' = \frac{1}{c^2} \frac{d^2}{dt^2} E
\]

in vacuum
Onward to radiation

\[ E'' = \frac{1}{c^2} \frac{d^2}{dt^2} E \]

in vacuum

A. The relationship for radiation involves the electric field \( E \) and magnetic field \( B \)

B. They are coupled

C. In space-time

D. With second derivative in space and in time.
Onward to radiation

\[ E'' = \frac{1}{c^2} \frac{d^2}{dt^2} E \]
in vacuum

A. The relationship for radiation involves the electric field (E) and magnetic field (B)
B. They are coupled
C. In space-time
D. With second derivative in space and in time.
E. Wiggles that move
Spacetime implications

\[ E'' = \frac{1}{c^2} \frac{d^2}{dt^2} E \]
Spacetime implications

\[ E'' = \frac{1}{c^2} \frac{d^2}{dt^2} E \]

6. Sines and cosines of time and space
Spacetime implications

\[ E'' = \frac{1}{c^2} \frac{d^2}{dt^2} E \]

6. Sines and cosines of time and space

7. \( \sin(kx \pm \omega t) \)
Spacetime implications

\[ E'' = \frac{1}{c^2} \frac{d^2}{dt^2} E \]

6. Sines and cosines of time and space
7. Sin(kx ± ωt)
8. Cos(kx ± ωt)
Spacetime implications

\[ E'' = \frac{1}{c^2} \frac{d^2}{dt^2} E \]

6. Sines and cosines of time and space
7. \( \sin(kx \pm \omega t) \)
8. \( \cos(kx \pm \omega t) \)
9. Kosher \( \omega \) and \( k \), if…
Spacetime implications

\[ E'' = \frac{1}{c^2} \frac{d^2}{dt^2} E \]

\[ E'' = -k^2 E \]

6. Sines and cosines of time and space

7. Sin(kx ± ωt)

8. Cos(kx ± ωt)

9. Kosher ω and k, if…
Spacetime implications

\[ E'' = \frac{1}{c^2} \frac{d^2}{dt^2} E \]

\[ E'' = -k^2 E \]

\[ \frac{d^2}{dt^2} E = -\omega^2 E \]

6. Sines and cosines of time and space

7. Sin(kx ± \omega t)

8. Cos(kx ± \omega t)

9. Kosher \omega and k, if…
Another brainiac question coming...

\[ E'' = -k^2 E \]

\[ \frac{d^2}{dt^2} E = -\omega^2 E \]
Space-time dependence of radiation and its speed of propagation

\[ c^2 = \frac{\omega^2}{k^2} \]

Speed of light couples the temporal and spatial dependences… a.k.a. frequency and wavelength!
What is this physical relationship to $c$?

\[ c^2 = \frac{\omega^2}{k^2} \]
What is this physical relationship to \( c \)?

A. Dispersion relation (fancy terminology)

\[
c^2 = \frac{\omega^2}{k^2}
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B. Refraction of light (transmission)

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✔ E.g., prisms disperse the colors of sunlight

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   - E.g., prisms disperse the colors of sunlight

C. Absorption of light

\[ c^2 = \frac{\omega^2}{k^2} \]
What is this physical relationship to c?

A. Dispersion relation (fancy terminology)

B. Refraction of light (transmission)
   ✓ E.g., prisms disperse the colors of sunlight

C. Absorption of light
   ✓ E.g., Superman, lead absorbs xrays.

\[ c^2 = \frac{\omega^2}{k^2} \]
Penetrating a material with radiation
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A. Inside material, the physics changes.
Penetrating a material with radiation

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B. Light moves more slowly.
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Penetrating a material with radiation

A. Inside material, the physics changes.
B. Light moves more slowly.
C. Energy is absorbed from E and B.
D. Heat flows, outer surfaces cool off
E. New spatial and temporal derivatives for E and B fields.
Penetrating a material with radiation

A. Inside material, the physics changes.
B. Light moves more slowly.
C. Energy is absorbed from E and B.
D. Heat flows, outer surfaces cool off.
E. New spatial and temporal derivatives for E and B fields.

» sine, cosine = NO GO!
Now a new derivatives relation must hold.
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F. The old derivatives relationship changes, i.e., for $E_z$...
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\[
E''_z - h^2 E_z = 0
\]

\[
E''_z = +h^2 E_z
\]
Now a new derivatives relation must hold.

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\]

G. New deriv. rel. means new functions do the work.

H. Hyperbolic functions, $\cosh(u)$ and $\sinh(u)$. 
Next Monday:

A. Radiation penetrating “cheese”
B. Quantum tunneling
C. Space-time structure