

# **Application of Series in Heat Transfer - transient heat conduction**

**By**

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**UCF EXCEL Applications of Calculus**



- **Part I – background and review of series  
(Monday 14, April 2008)**

- 1. Taylor and Maclaurin series (section 12.10)**
- 2. Fourier series**

- **Part II – applications (Monday 21, April 2008)**

- 1. Transient heat conduction**
- 2. Finite Difference**

## Background and review of series

- 1. Taylor series (section 12.10)** - developed through the works of J. Gregory, L. Euler, B. Taylor and C. Maclaurin ~ 18<sup>th</sup> Century  
- named after B. Taylor

- Can be used represent any function,  $f(x)$ , that is infinitely differentiable about a point,  $x_0$ , called the expansion point, and the series is

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + f''(x_0)\frac{(x-x_0)^2}{2!} + \dots = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)(x-x_0)^n}{n!}$$

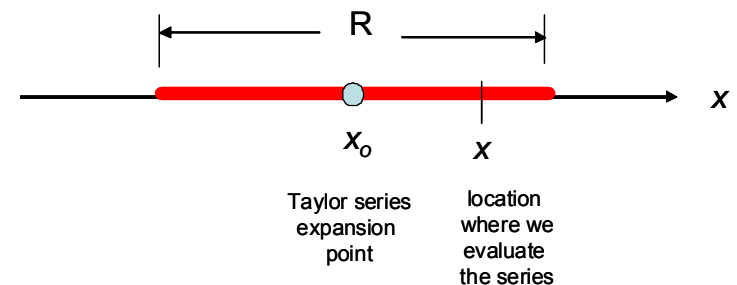
where,

$f'$  first derivative of  $f(x)$

$f''$  second derivative of  $f(x)$

⋮

$f^{(n)}$   $n$ -th derivative of  $f(x)$



- The location,  $x$ , where the series is evaluated can be taken anywhere within the radius of convergence of the series in order to yield the correct value for  $f(x)$ .

# Application of Series in Heat Transfer: transient heat conduction



**James Gregory** (1638-1675) Scottish mathematician and astronomer at the University of St. Andrews in 1667 publishes series expansions for  $\sin(x)$ ,  $\cos(x)$ ,  $\arcsin(x)$  and  $\arccos(x)$  that turn out to be Maclaurin series for these functions.



**Brook Taylor** (1685-1731) English Mathematician, Cambridge University. Published his discoveries on series and what we call the Taylor theorem that according to Joseph Louis Lagrange was “the main foundation of differential calculus” in *Methodus Incrementorum Directa et Inversa* (1715).



**Colin Maclaurin** (1698-1746), Scottish mathematician at the University of Edinburgh. Published series for trigonometric functions that now bear his name and popularizes them in his Calculus book *Treatise of Fluxions* (1742).

Note: earlier **Mādhava of Sangamagrama** (14<sup>th</sup> Century), (1350-1425) an Indian mathematician and astronomer had also discovered series for some trigonometric functions

# Application of Series in Heat Transfer: transient heat conduction

● **Leonhard Paul Euler** (1707-1783) a pioneering Swiss mathematician and physicist who made seminal discoveries in all branches of pure and applied mathematics and Calculus. He introduced much of the modern mathematical terminology and notation. He made major contributions to solid and fluid mechanics, optics and astronomy. Euler is considered to be the preeminent mathematician of the 18th century and one of the greatest of all time. He was blind for nearly half of his life and, despite that affliction, he was such a prolific scientific author that his discoveries are still being published. Pierre-Simon Laplace said of Euler “Read Euler, read Euler, he is the master us all.”



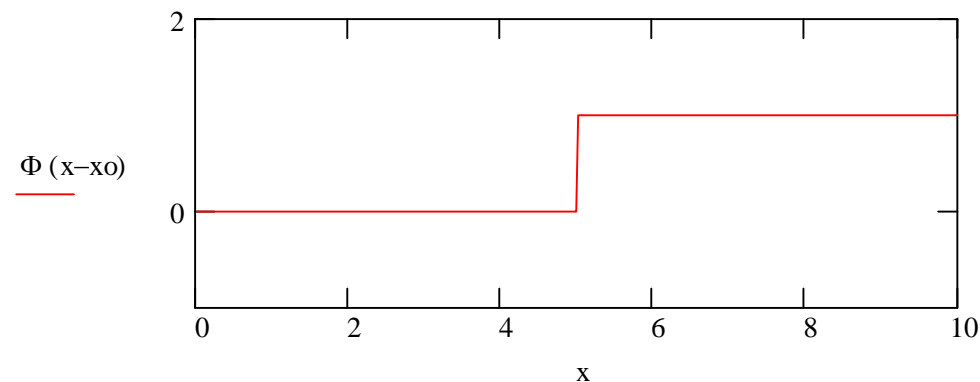
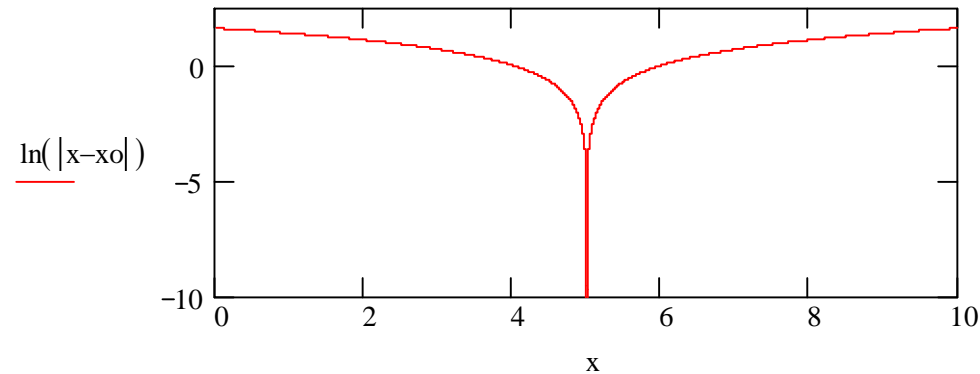
● In your studies you'll encounter his contributions in nearly every subject and when referring to Euler's Formula you will wonder which one? One of his most famous often called the Euler Formula is:

$$e^{i\varphi} = \cos \varphi + i \sin \varphi$$

● He discovered the Maclaurin series for  $e^x$ : 
$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \lim_{n \rightarrow \infty} \left( \frac{1}{0!} + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} \right)$$

# Application of Series in Heat Transfer: transient heat conduction

- infinitely differentiable about a point,  $x_0$ , means the function can't be “badly behaved” at  $x_0$
- examples of functions that are “badly behaved” and not differentiable at  $x_0 = 5$  and, therefore, cannot be expanded in a Taylor series about that point.



# Application of Series in Heat Transfer: transient heat conduction

- The radius of convergence of the series can be found using the ratio test (section 12.6).
- Practically, the radius of convergence is from the point  $x_0$  to the nearest point where  $f(x)$  is no longer well behaved.

**Example:**  $f(x) = \ln(x)$  expanded about  $x_0 = 2$

- Taking the derivatives and evaluating at  $x_0 = 2$

$$f(x) = \ln(x)$$

$$f'(x) = \frac{1}{x}$$

$$f''(x) = -\frac{1}{x^2}$$

$$f'''(x) = \frac{2}{x^3}$$

$$f^{(4)}(x) = -\frac{2 \cdot 3}{x^4}$$

$$f^{(5)}(x) = \frac{2 \cdot 3 \cdot 4}{x^5}$$

⋮

⋮

⋮

$$f^{(n)}(x) = (-1)^{n+1} \frac{(n-1)!}{x^n}$$

$$f(2) = \ln(2)$$

$$f'(2) = \frac{1}{2}$$

$$f''(2) = -\frac{1}{2^2}$$

$$f'''(2) = \frac{2}{2^3}$$

$$f^{(4)}(2) = -\frac{2 \cdot 3}{2^4}$$

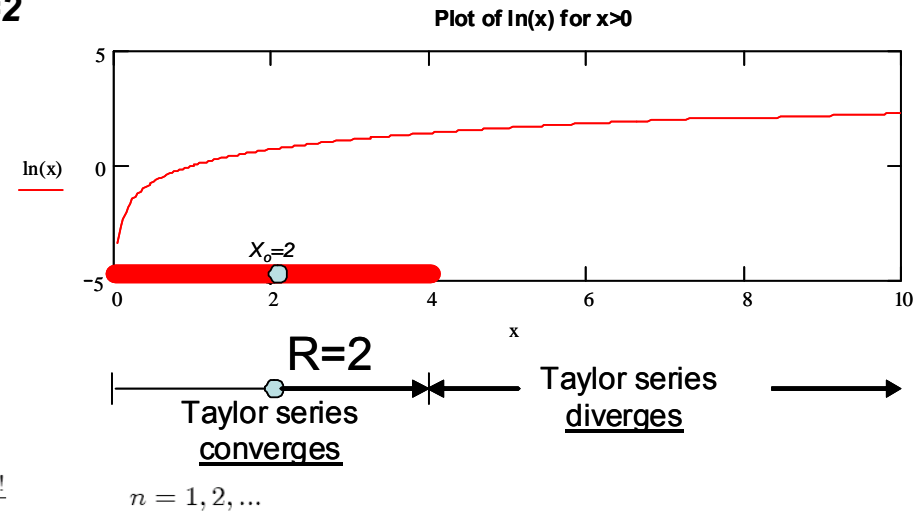
$$f^{(5)}(2) = \frac{2 \cdot 3 \cdot 4}{2^5}$$

⋮

⋮

⋮

$$f^{(n)}(2) = (-1)^{n+1} \frac{(n-1)!}{2^n}$$



- Introducing into the Taylor series definition, we have the series for  $\ln(x)$  about  $x_0 = 2$

$$\begin{aligned} \ln(x) &= \ln(2) + \sum_{n=1}^{\infty} \left[ (-1)^{n+1} \frac{(n-1)!}{2^n} \right] \frac{(x-2)^n}{n!} \\ &= \ln(2) + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-2)^n}{n 2^n} \end{aligned}$$

....and since  $\ln(x)$  is badly behaved (goes to  $-\infty$ ) at  $x=0$ , the series has radius of convergence  $R=2$  or  $x$  can be taken anywhere  $0 < x < 4$

- Let's compute this series with MATHCAD

# Application of Series in Heat Transfer: transient heat conduction

Expand  $\ln(x)$  about  $x_0=2$

$$f(x, N) := \ln(2) + \sum_{n=1}^N \left[ (-1)^{n+1} \cdot \frac{(x-2)^n}{n \cdot 2^n} \right]$$

Evaluate series at point

$$x_c := 2.5$$

Take M-terms in the series

$$M := 10$$

$$N := 1, 2..M$$

**X=2.5 is inside radius of convergence**  
**→ Series converges**

Number of Taylor series terms in series for  $\ln(x)$  about  $x_0=2$

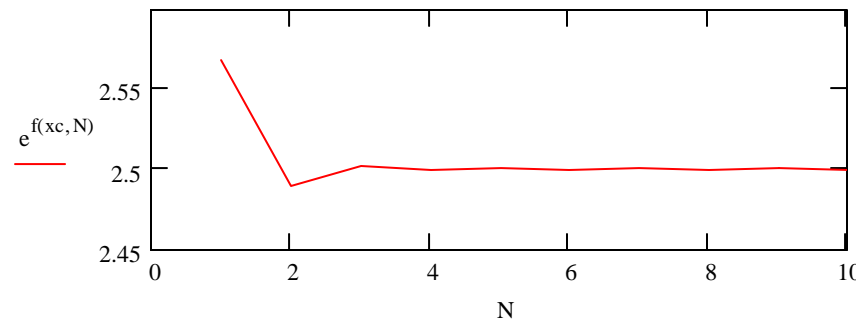
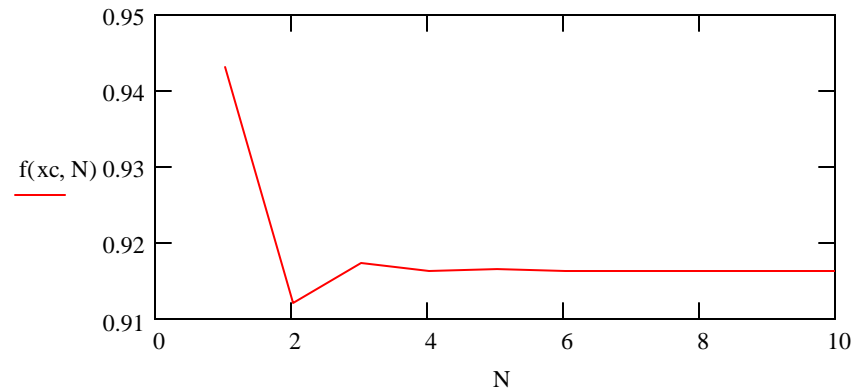
N =	$f(x_c, N) =$
1	0.9431471806
2	0.9118971806
3	0.9171055139
4	0.9161289514
5	0.9163242639
6	0.9162835738
7	0.9162922931
8	0.9162903857
9	0.9162908096
10	0.9162907142

take anti-log of result to check

$$e^{f(x_c, N)} =$$

2.568
2.489
2.502
2.5
2.5
2.5
2.5
2.5
2.5
2.5
2.5

$$e^{f(x_c, M)} = 2.4999999559$$



- **Observation:**  $x=2.5$  close to expansion point and series converges quickly (takes very few terms)



# Application of Series in Heat Transfer: transient heat conduction

Expand  $\ln(x)$  about  $x_0=2$

$$f(x, N) := \ln(2) + \sum_{n=1}^N \left[ (-1)^{n+1} \cdot \frac{(x-2)^n}{n \cdot 2^n} \right]$$

Evaluate series at point

$$x_c := 3.9$$

Take M-terms in the series

$$M := 100$$

$$N := 1, 2..M$$

**X=3.9 is inside radius of convergence**  
**→ Series converges**

Number of Taylor series terms in series for  $\ln(x)$  about  $x_0=2$

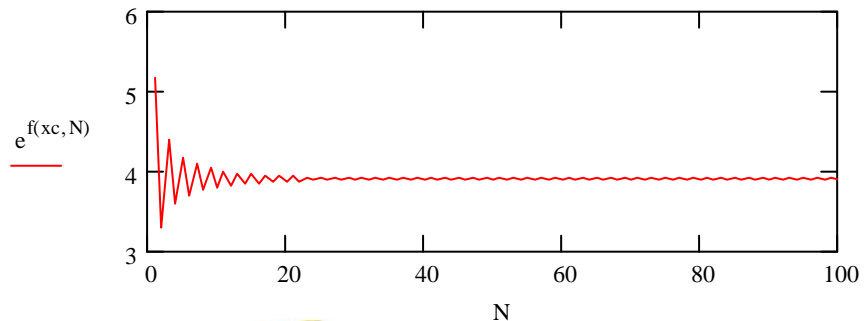
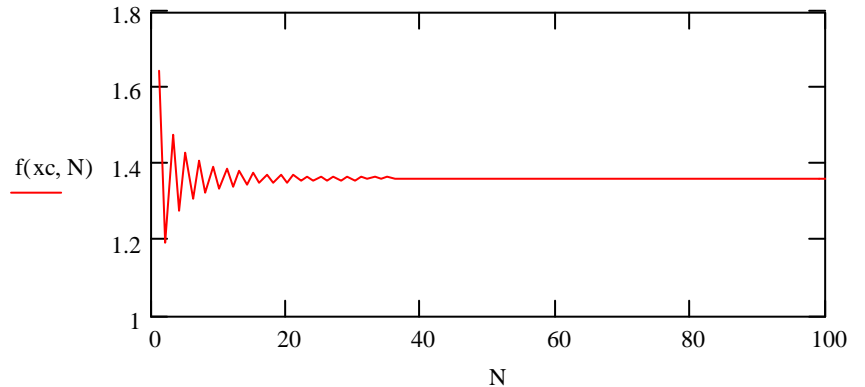
N =	f(xc, N) =
1	1.6431471806
2	1.1918971806
3	1.4776888472
4	1.2740622847
5	1.4288184722
6	1.3063031571
7	1.406065628
8	1.3231380741
9	1.3931657863
10	1.3332920923
11	1.3850011916
12	1.3399711843
13	1.3794590369
14	1.3446251098
15	1.3755111918
16	1.348003275
17	

take anti-log of result to check

$$e^{f(xc, N)} =$$

5.171
3.293
4.383
3.575
4.174
3.692
4.08
3.755
4.028
3.794
3.995
3.819
3.973
3.837
3.957
3.85
3.946

$$e^{f(xc, M)} = 3.8998880882$$



- **Observation:**  $x=3.9$  far from expansion point and series converges slowly (takes many terms)

# Application of Series in Heat Transfer: transient heat conduction

Expand  $\ln(x)$  about  $x_0=2$

$$f(x, N) := \ln(2) + \sum_{n=1}^N \left[ \frac{(-1)^{n+1} \cdot (x-2)^n}{n \cdot 2^n} \right]$$

Evaluate series at point

$$xc := 4.1$$

Take M-terms in the series

$$M := 150$$

$$N := 1, 2, \dots, M$$

**X=4.1 is outside radius of convergence**  
**→ Series diverges**

Number of Taylor series terms in series for  $\ln(x)$  about  $x_0=2$

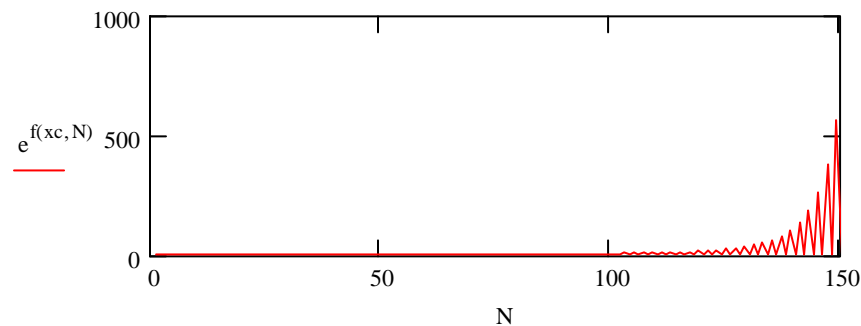
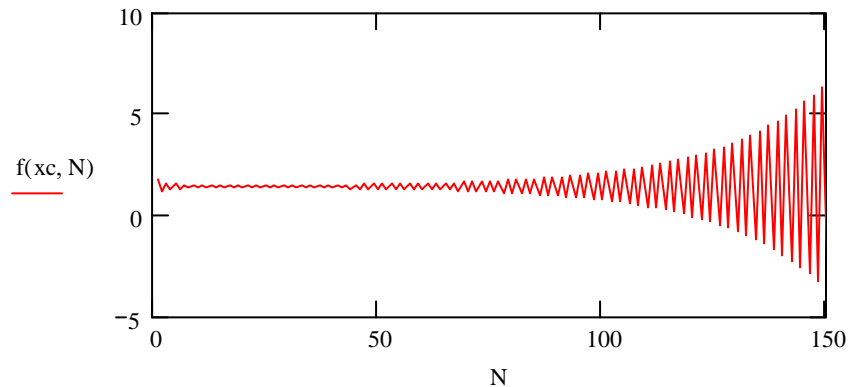
N =	f(xc, N) =
1	1.7431471806
2	1.1918971806
3	1.5777721806
4	1.2738956181
5	1.5291519306
6	1.3058026571
7	1.5068170032
8	1.3221350727
9	1.4945048745
10	1.3316154118
11	1.487100808
12	1.3374461142
13	1.4824960482
14	1.3410723625
15	1.4796675745
16	1.3432379127
17	

take anti-log of result to check

$$e^{f(xc, N)} =$$

5.715
3.293
4.844
3.575
4.614
3.691
4.512
3.751
4.457
3.787
4.424
3.809
4.404
3.823
4.391
3.831
4.384

$$e^{f(xc, M)} = 0.0241986898$$



# Application of Series in Heat Transfer: transient heat conduction

- As we just saw in practice, when we compute using series, we can never add an infinite number of terms, so we truncate the series after  $N$ -terms (stop after adding  $N$ -terms) and, from the Taylor remainder theorem, we have an approximation of the function  $f(x)$  as

$$f(x) = \underbrace{\sum_{n=0}^N \frac{f^{(n)}(x_0)(x-x_0)^n}{n!}}_{\substack{\text{terms we compute} \\ \text{and add to approximate} \\ \text{the function } f(x)}} + \underbrace{\frac{f^{(n+1)}(\xi)(x-x_0)^{n+1}}{(n+1)!}}_{\substack{\text{the term we truncate} \\ \text{and drop from our} \\ \text{computation of } f(x) \text{ is} \\ \text{the truncation error (TE)}}$$

↑
↑  
 Taylor polynomial:  $T_n(x)$       Remainder:  $R_n(x)$

**Note:** this expression for  $R_n(x)$  is called the *Lagrange form of the remainder*. There are two other alternative forms: *integral form of the remainder* and *Taylor's inequality*.

- While we do not know  $\xi$  exactly, we do know at least that its location is somewhere  $x_0 < \xi < x$
- More importantly, although we do not know the truncation error (TE) exactly, we know at least that it is proportional to  $(x-x_0)^{n+1}$  [*helps explain rate of convergence*].
- How many terms do we take in the series? we take enough terms such that adding more terms will not make a significant change in the value of the sum we compute to approximate  $f(x)$ .
- For converging alternating series can rely on the *Alternating Series Estimation Theorem* (section 12.5)

# Application of Series in Heat Transfer: transient heat conduction

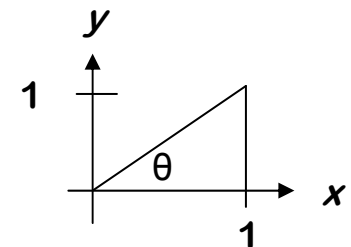
- The Maclaurin series is a Taylor series expanded about the particular expansion point  $x_0 = 0$ , that is

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + \dots = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

- What the big deal? “discovered” independently and the advantage is that if a function behaves well everywhere on the real axis (and can be expanded about any point) then the special point  $x=0$  gives the easiest form of the series to differentiated and to compute.

**Example:** computing  $\pi$  with the Maclaurin series for  $\tan^{-1}(\theta)$

$f^{(0)}(x) = \tan^{-1}(x)$	$f^{(0)}(0) = ?$
$f^{(1)}(x) = \frac{1}{x^2+1} = (x^2 + 1)^{-1}$	$f^{(1)}(0) = 1$
$f^{(2)}(x) = -2x(x^2 + 1)^{-2}$	$f^{(2)}(0) = 0$
$f^{(3)}(x) = 8x^2(x^2 + 1)^{-3} - 2(x^2 + 1)^{-2}$	$f^{(3)}(0) = -2 = -2!$
$f^{(4)}(x) = -48x^3(x^2 + 1)^{-4} + 24x(x^2 + 1)^{-3}$	$f^{(4)}(0) = 0$
$f^{(5)}(x) = 384x^4(x^2 + 1)^{-5} - 288x^3(x^2 + 1)^{-4}$	$f^{(5)}(0) = 24 = 4!$
$+ 24(x^2 + 1)^{-3}$	
⋮	⋮
⋮	⋮
⋮	⋮



$$\tan(\theta) = \frac{\pi}{4} = 1$$

$$\therefore \tan^{-1}(1) = \frac{\pi}{4}$$

# Application of Series in Heat Transfer: transient heat conduction

- so that the Maclaurin series for  $\tan^{-1}(x)$ , also called Gregory's series, is

$$\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} \dots = \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

- and for the specific case that  $x=1$  and  $\tan^{-1}(1) = \pi/4$ , then

$$\frac{\pi}{4} = \tan^{-1}(1) = 1 - \frac{1}{3} + \frac{1}{5} \dots = \sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1}$$

- or a way to compute  $\pi$  (called the Leibniz formula for  $\pi$ ) is

$$\pi = 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1}$$

- **Let's compute with MATHCAD**

Note: although the series converges on the interval,  $-1 \leq x \leq 1$ , from our previous experience we expect, that since we are interested to evaluate the series at  $x=1$ , slow convergence.

# Application of Series in Heat Transfer: transient heat conduction

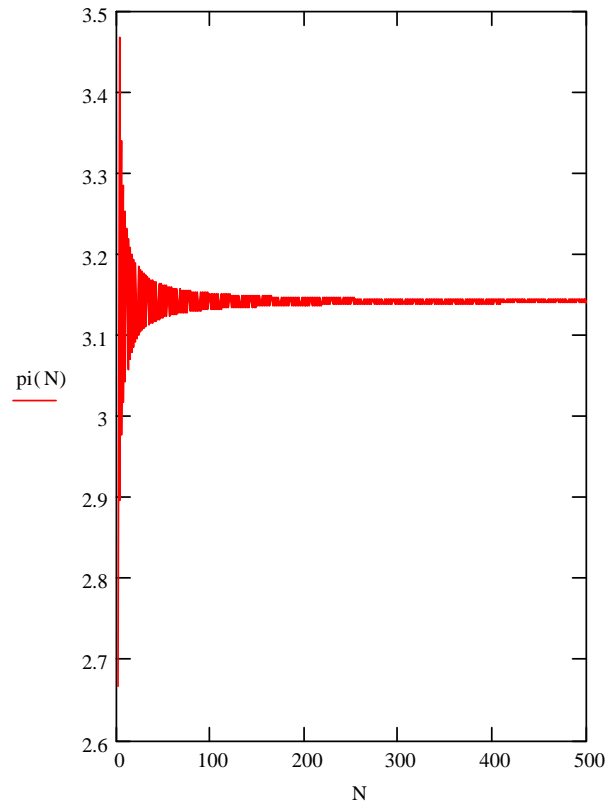
Leibniz formula: 
$$\text{pi}(N) := 4 \cdot \sum_{n=0}^N \frac{(-1)^n}{2n+1}$$

Take M-terms  $M := 500$

$N := 1, 2.. M$

$\text{pi}(M) = 3.1435886596 \quad \pi = 3.1415926536$

N =	pi(N) =
1	2.667
2	3.467
3	2.895
4	3.34
5	2.976
6	3.284
7	3.017
8	3.252
9	3.042
10	3.232
11	3.058
12	3.218
13	3.07
14	3.208
15	3.079
16	3.2
17	3.086
18	3.194
19	3.092
20	3.189
21	3.096
22	3.185
23	3.1
24	3.182
25	3.103

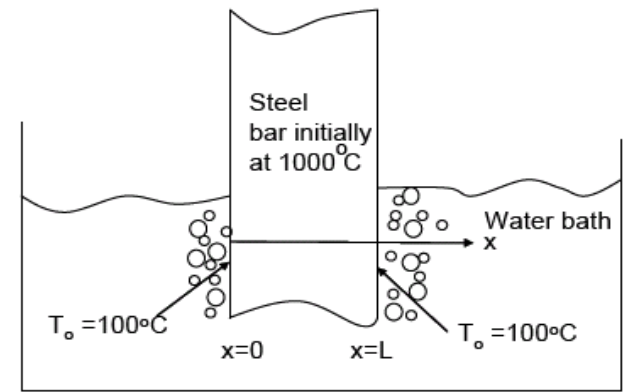


# Application of Series in Heat Transfer: transient heat conduction

- OK, that's great, but how do we utilize such series in practice?
- In part II, we will consider practical examples of the application of Taylor and Maclaurin series to the determination of the temperature of a hot metal bar being quenched (rapidly cooled) in water bath and the applications of Taylor series in computational methods in heat transfer and fluid flow (finite difference methods).

- The solution to certain heat conduction problems (EML 4142 – Heat Transfer) in which we seek the temperature as a function of time and space,  $T(x,t)$ , involves what is called the error function,  $erf(x)$ , which is defined by the integral

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-x^2} dx$$



Quenching of a metal bar

- This integral has no closed form solution, that is we cannot find a formula, a method, or a combination of tricks from calculus to integrate and find an explicit expression for the integral.
- How are we to then compute the temperature?
- We can utilize the Maclaurin series for  $e^{-x^2}$  into the integral and integrate term by term (section 12.9).

# Application of Series in Heat Transfer: transient heat conduction

- The Maclaurin series for  $e^{\pm x}$  is

$$e^{\pm x} = 1 \pm x + \frac{x^2}{2!} \pm \frac{x^3}{3!} + \frac{x^4}{4!} \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$$

which converges on  $-\infty < x < +\infty$ .

- Utilizing this result we can express  $e^{-x^2}$  as a series that converges also on  $-\infty < x < +\infty$  and therefore can be integrated term by term and will yield a series that will converge for any value of  $x$ .



# Application of Series in Heat Transfer: transient heat conduction

- Integrating term by term

$$\begin{aligned} \operatorname{erf}(x) &= \frac{2}{\sqrt{\pi}} \int_0^x e^{-x^2} dx \\ &= \frac{2}{\sqrt{\pi}} \int_0^x \left[ 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} + \dots \right] dx \\ &= \frac{2}{\sqrt{\pi}} \left[ x - \frac{x^3}{3} + \frac{x^5}{2!4} - \frac{x^7}{3!7} + \frac{x^9}{4!9} + \dots \right] \\ &= \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(2n+1)} \end{aligned}$$

- So that the error function can be written as the series:

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(2n+1)}$$

- You will also encounter this function in statistics (STA 3032) as it is also called the probability function, and, in statistics, you will likely be told to use look-up tables to evaluate this function. After this discussion, you will know another way than to use a table.
- Let's compute the error function series for various values of  $x$  using MATHCAD

# Application of Series in Heat Transfer: transient heat conduction

**Error function** 
$$\text{Erf}(x, N) := \frac{2}{\sqrt{\pi}} \cdot \sum_{n=0}^N \frac{(-1)^n \cdot x^{2n+1}}{n! \cdot (2n+1)}$$

$x_0 := 0.2$

$M := 10$

$N := 1, 2..M$

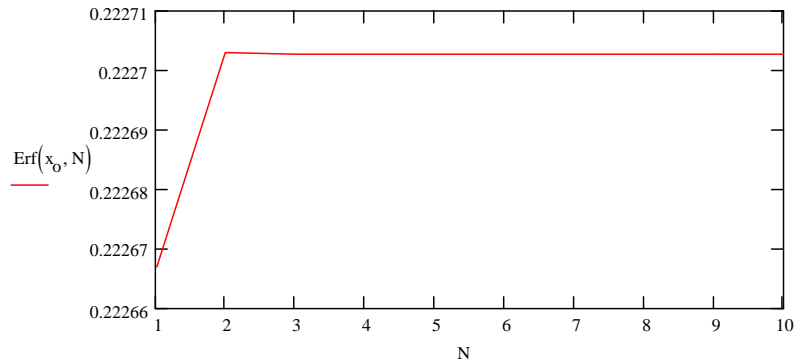
$n := 0, 2..M$

$nn := 1, 3..M$

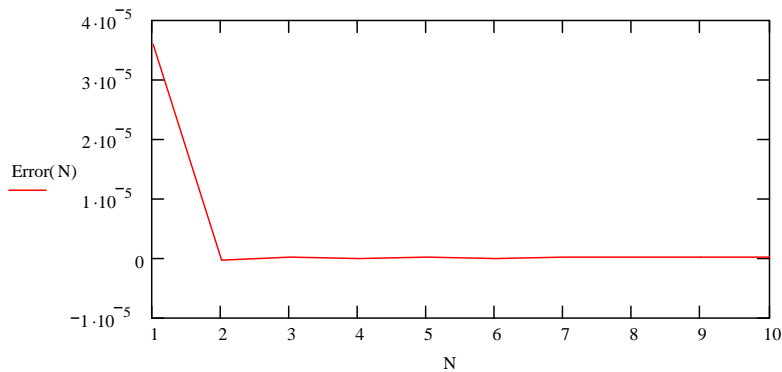
$\text{erf}(x_0) = 0.222702589210479$

$\text{Erf}(x_0, M) = 0.222702589210478$

$\text{Error}(N) := \text{erf}(x_0) - \text{Erf}(x_0, N)$



N =	Erf(x <sub>0</sub> , N) =	Error(N) =
1	0.2226668	3.577 · 10 <sup>-5</sup>
2	0.2227029	-3.412 · 10 <sup>-7</sup>
3	0.2227026	2.657 · 10 <sup>-9</sup>
4	0.2227026	-1.741 · 10 <sup>-11</sup>
5	0.2227026	9.837 · 10 <sup>-14</sup>
6	0.2227026	0
7	0.2227026	0
8	0.2227026	0
9	0.2227026	0
10	0.2227026	0



# Application of Series in Heat Transfer: transient heat conduction

**Error function** 
$$\text{Erf}(x, N) := \frac{2}{\sqrt{\pi}} \cdot \sum_{n=0}^N \frac{(-1)^n \cdot x^{2n+1}}{n! \cdot (2n+1)}$$

$x_0 := 1.5$

$M := 10$

$N := 1, 2, \dots, M$

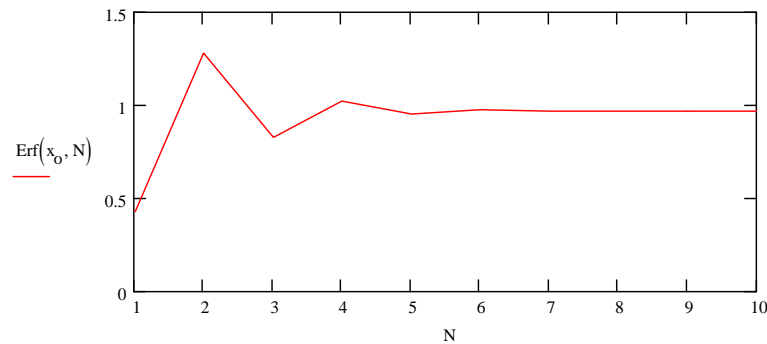
$n := 0, 2, \dots, M$

$nn := 1, 3, \dots, M$

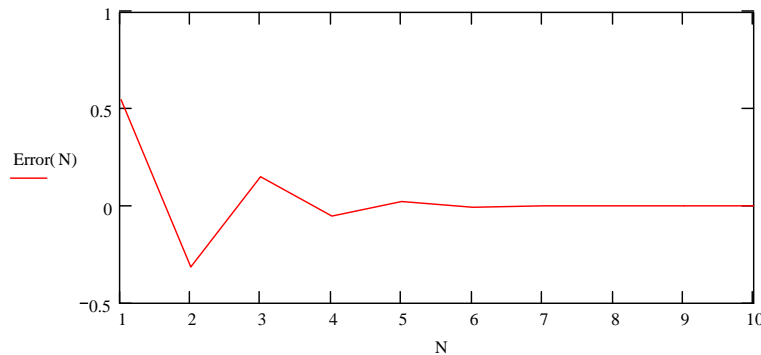
$\text{erf}(x_0) = 0.966105146475311$

$\text{Erf}(x_0, M) = 0.966116891809457$

$\text{Error}(N) := \text{erf}(x_0) - \text{Erf}(x_0, N)$



N =	Erf(x <sub>0</sub> , N) =	Error(N) =
1	0.4231422	0.543
2	1.2800051	-0.314
3	0.8209714	0.145
4	1.0217987	-0.056
5	0.9478577	0.018
6	0.9713197	-5.215 · 10 <sup>-3</sup>
7	0.9647839	1.321 · 10 <sup>-3</sup>
8	0.9664058	-3.007 · 10 <sup>-4</sup>
9	0.966043	6.211 · 10 <sup>-5</sup>
10	0.9661169	-1.175 · 10 <sup>-5</sup>



# Application of Series in Heat Transfer: transient heat conduction

**Error function** 
$$\text{Erf}(x, N) := \frac{2}{\sqrt{\pi}} \cdot \sum_{n=0}^N \frac{(-1)^n \cdot x^{2n+1}}{n! \cdot (2n+1)}$$

$x_0 := 2.5$

$M := 50$

$N := 1, 2, \dots, M$

$n := 0, 2, \dots, M$

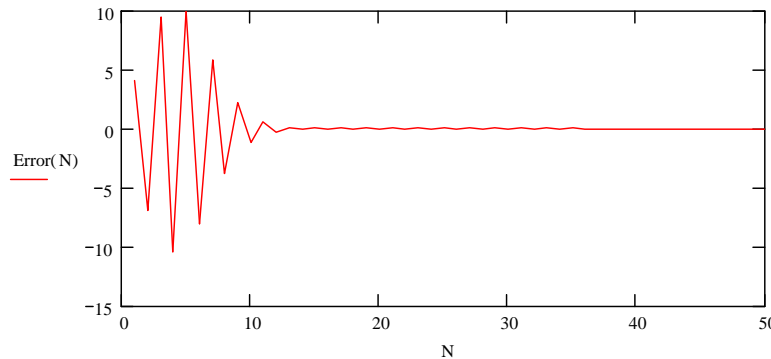
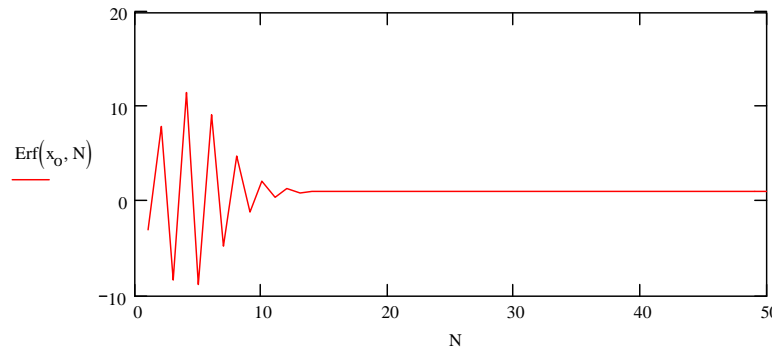
$nn := 1, 3, \dots, M$

$\text{erf}(x_0) = 0.999593047982555$

$\text{Erf}(x_0, M) = 0.999593047982557$

$\text{Error}(N) := \text{erf}(x_0) - \text{Erf}(x_0, N)$

$\text{Erf}(x_0, M) = 1$



N =	Erf(x <sub>0</sub> , N) =	Error(N) =
1	-3.0560269	4.056
2	7.9633009	-6.964
3	-8.4345083	9.434
4	11.4933848	-10.494
5	-8.887415	9.887
6	9.076431	-8.077
7	-4.8241641	5.824
8	4.7580476	-3.758
9	-1.195812	2.195
10	2.1709538	-1.171
11	0.4243609	0.575
12	1.26127	-0.262
13	0.8887143	0.111
14	1.0435635	-0.044
15	0.9832056	0.016
16	1.005354	-5.761 · 10 <sup>-3</sup>

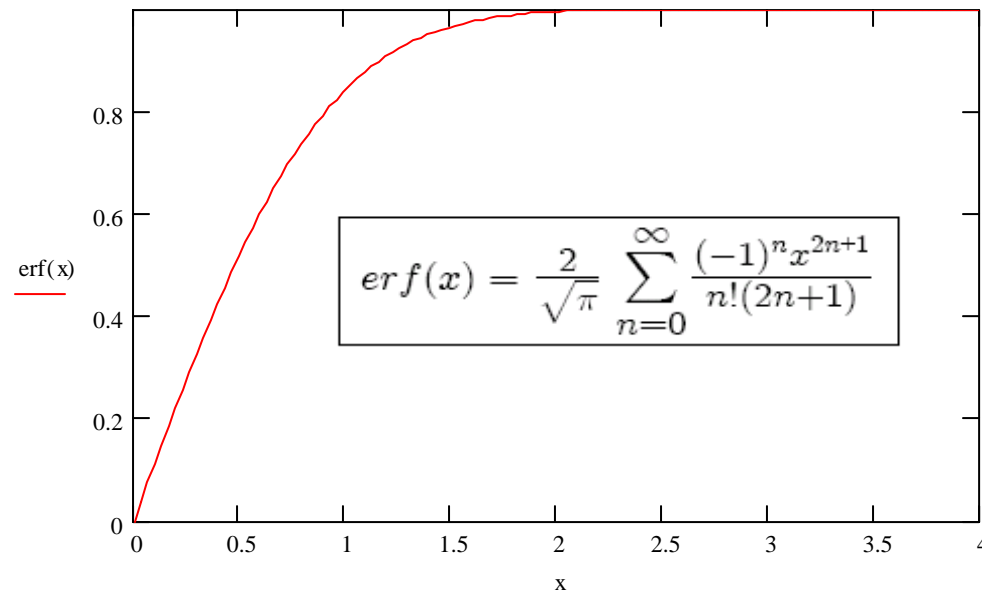
# Application of Series in Heat Transfer: transient heat conduction

## The error function of x: definition and series

$$\operatorname{erf}(0.5) = 0.52$$

$$\operatorname{erf}(2) = 0.995$$

$$\operatorname{erf}(2.5) = 1$$



$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-x^2} dx = \frac{2}{\sqrt{\pi}} \left[ x - \frac{x^3}{3} + \frac{x^5}{2!4} - \frac{x^7}{3!7} + \frac{x^9}{4!9} + \dots \right]$$

# Application of Series in Heat Transfer: transient heat conduction

**2. Fourier series** - due to the work of Jean-Baptiste Fourier (1768-1830) and named after him  
- L. Euler and Jacob Bernoulli also had made early discoveries summing certain sine and cosine series for specific functions.

- A function  $f(x)$  can be represented on  $[-L,L]$  as a sum of sines and cosines, the Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right)$$

where the coefficients in the series are given by:

$$a_0 = \frac{1}{2L} \int_{-L}^L f(x) dx$$

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi}{L}x\right) dx$$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi}{L}x\right) dx$$

provide the amplitude for every trigonometric wave: applications in acoustics, vibrations, heat transfer, signal processing, imaging, ....

# Application of Series in Heat Transfer: transient heat conduction

**Jean-Baptiste Joseph Fourier** (1768-1830) who lived in the days of Napoleon whom he accompanied in his 1798 campaigns to Egypt and served as governor of Lower Egypt under French rule.

He later became Prefect of the Department of Isere in South-Western France where he carried out the famous experimental and analytical research that laid the foundations of heat transfer.

It was a great accomplishment by Fourier to show that any function, with some reasonable degree to continuity, could be expressed in terms of summations of trigonometric functions that now bear his name: Fourier series. Actually, some of the greatest mathematicians and physicists of his time were not convinced of this and made his life a bit miserable.

He published these results and the foundations of heat transfer in solids in his famous 1822 treatise *Théorie Analytique de la Chaleur* (the Analytical Theory of Heat) which got him elected as a member of the French National Academy of Science (with special mention for *lack of rigor ...* are the French tough or what!), of which he was later to be the Permanent Secretary.

Fourier is also credited with the discovery in 1824 that gases in the atmosphere might increase the surface temperature of the Earth: **the greenhouse effect**.



- Mathematician, physicist and historian
- Ecole Normale and Ecole Polytechnique

# Application of Series in Heat Transfer: transient heat conduction

The **Bernoullis** were a family of Swiss traders and scholars. The founder of the family, Nicolaus Bernoulli, immigrated to Basel from Flanders in the 16<sup>th</sup> Century. The Bernoulli family has produced many notable mathematicians, scientists, philosophers and artists, and in particular several of the most well-known mathematician the 18th century: the brothers Jacob and Johann Bernoulli and Daniel Bernoulli.

Nicolaus Bernoulli (1623-1708)

Jacob Bernoulli (1654–1705; also Jacques): Bernoulli numbers and  
Nicolaus Bernoulli (1662–1716) separation of variables

Nicloaus I Bernoulli (1687–1759)

Johann Bernoulli (1667–1748): developed L'Hopital's rule & sold it to him

Nicolaus II Bernoulli (1695–1726)

Daniel Bernoulli (1700–1782): Bernoulli Equation

Johann Bernoulli II (1710–1790)

Johann Bernoulli II (1744–1807)

Daniel II Bernoulli (1751–1834)

Jacob Bernoulli II (1759–1789)



The brothers: *Jacob and Johann*



Johann's son  
*Daniel*

$$p + \frac{1}{2}\rho V^2 + gh = \text{constant}$$



# Application of Series in Heat Transfer: transient heat conduction

- How does this series work? Lets examine the terms

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right)$$

1. the mean value:

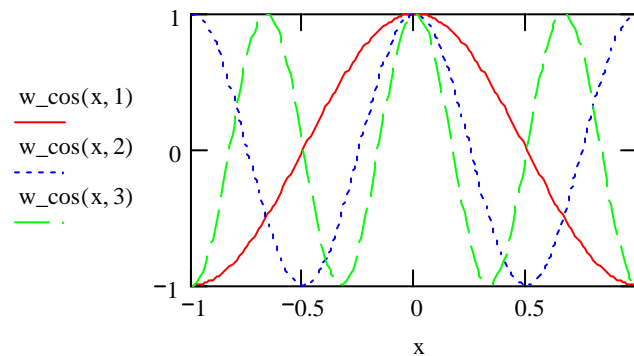
$$a_0 = \frac{1}{2L} \int_{-L}^L f(x) dx$$

2. amplitude of cosine waves:

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi}{L}x\right) dx$$

also called the  
discrete Fourier  
cosine transform

$$w_{\cos}(x, n) := \cos\left(\frac{n \cdot \pi}{L} \cdot x\right)$$



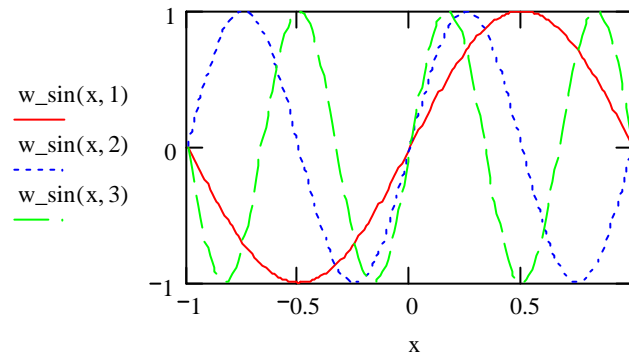
# Application of Series in Heat Transfer: transient heat conduction

3. amplitude of sine waves:

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi}{L}x\right) dx$$

also called the  
discrete Fourier  
sine transform

$$w_{\sin}(x, n) := \sin\left(\frac{n \cdot \pi}{L} \cdot x\right)$$



- J.B Fourier, 1822 treatise *Théorie Analytique de la Chaleur*
- J.P. Dirichlet (1805-1859) defines the **Dirichlet conditions** for  $f(x)$  to be represented by a Fourier series as well as the behavior of the Fourier series of  $f(x)$ :
  1.  $f(x)$  must be a piece-wise regular real-valued function defined on some interval, say  $[-L, L]$  or  $[0, L]$
  2.  $f(x)$  has only a finite number of discontinuities and extrema in that interval
  3. then Fourier series of this function converges to  $f(x)$  where  $f(x)$  is continuous **and** to the arithmetic mean of the limit to the left and limit to the right of the function  $f(x)$  at a point where  $f(x)$  is discontinuous.



Johann Peter Gustav Lejeune Dirichlet

# Application of Series in Heat Transfer: transient heat conduction

EXAMPLE:

$$f(x) = x \quad x \in [0, L]$$

- Calculate the Fourier series coefficients:

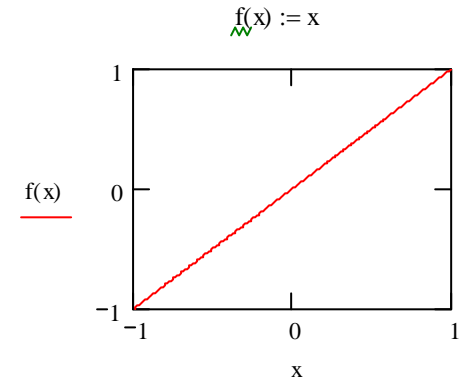
$$a_0 := \frac{2}{L} \int_{-L}^L f(x) dx$$

$$a_0 = 0$$

mean value of  $f(x)$   
on  $[-L, L]$

$$a_n := \frac{1}{L} \int_{-L}^L f(x) \cdot \cos\left(\frac{n \cdot \pi}{L} \cdot x\right) dx$$

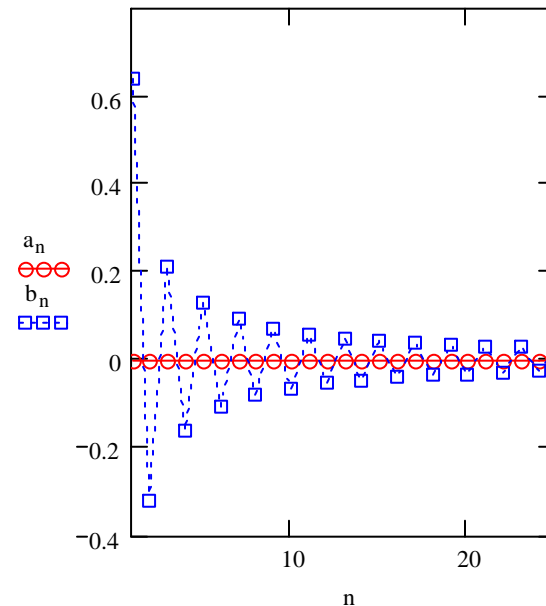
$$b_n := \frac{1}{L} \int_{-L}^L f(x) \cdot \sin\left(\frac{n \cdot \pi}{L} \cdot x\right) dx$$



- Let's compute this series with MATHCAD

# Application of Series in Heat Transfer: transient heat conduction

n =	a <sub>n</sub> =	b <sub>n</sub> =
1	0	0.637
2	0	-0.318
3	0	0.212
4	0	-0.159
5	0	0.127
6	0	-0.106
7	0	0.091
8	0	-0.08
9	0	0.071
10	0	-0.064
11	0	0.058
12	0	-0.053
13	0	0.049
14	0	-0.045
15	0	0.042
16	0	-0.04



- All the a<sub>n</sub>'s are zero!! actually, we already know this....why?

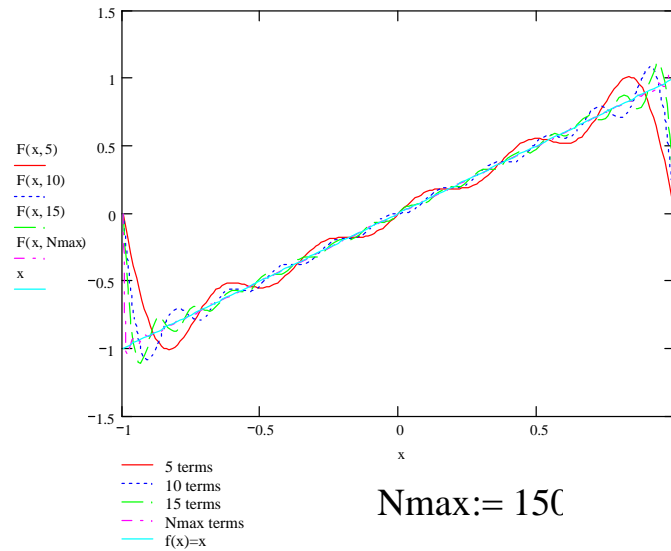
$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi}{L}x\right) dx \quad \text{and } f(x)=x$$

- Integral of odd function (cos even, x odd and xcos(x) is odd) between symmetric limits=0!

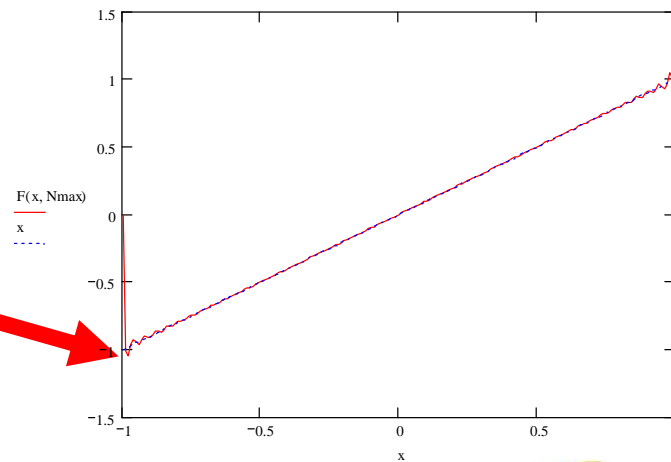
# Application of Series in Heat Transfer: transient heat conduction

- As we add more terms series converges

$$f(x, N) := a_0 + \sum_{n=1}^N \left( a_n \cos\left(n \frac{\pi}{L} x\right) + b_n \sin\left(n \frac{\pi}{L} x\right) \right)$$



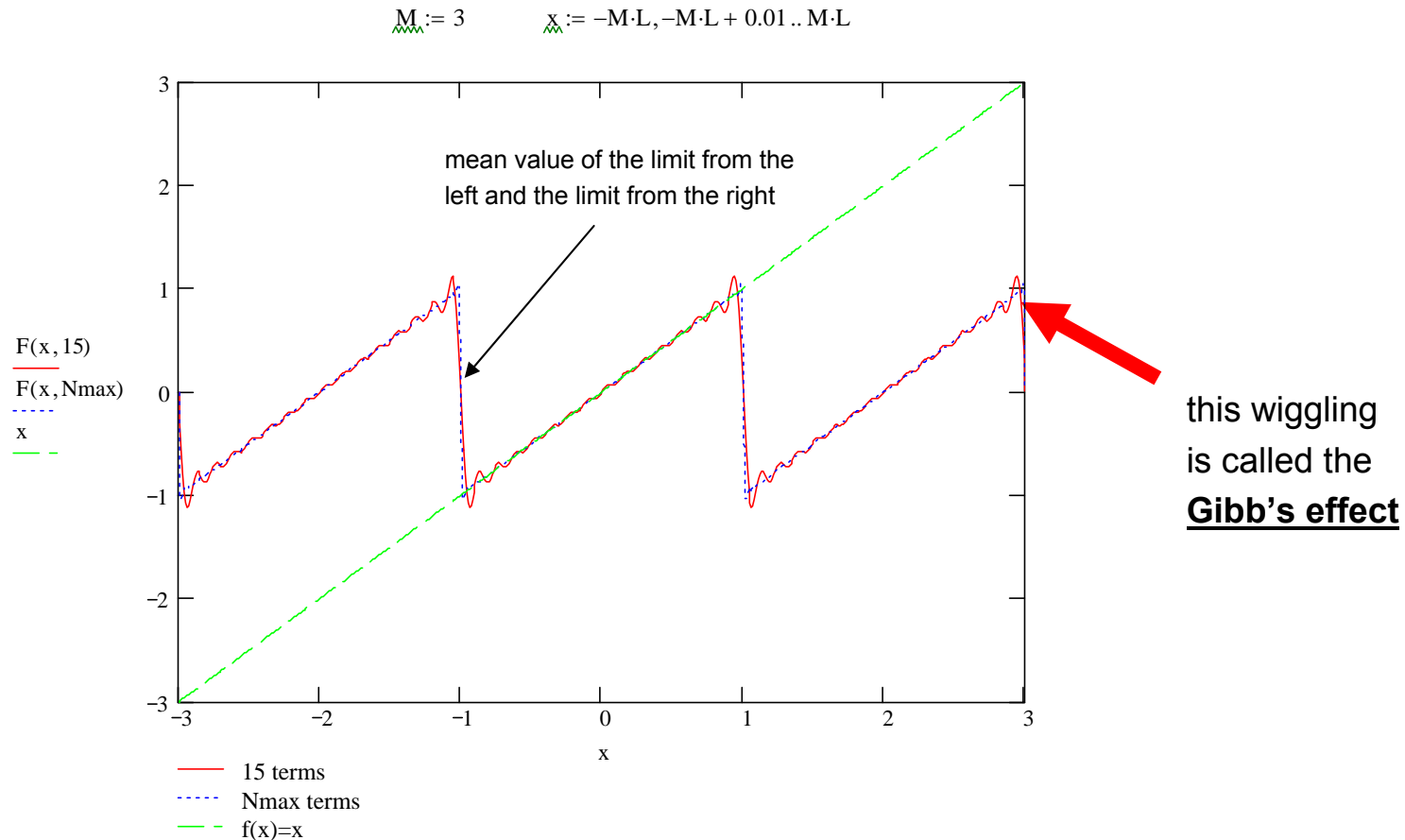
what's going on here?



what's going on here?

# Application of Series in Heat Transfer: transient heat conduction

- Since we represent a function with sum of periodic functions result is periodic!!! and at endpoints converges to mean value from limit to the left and limit to the right: ol' J.P. (Dirichlet) was right !!!



# Application of Series in Heat Transfer: transient heat conduction

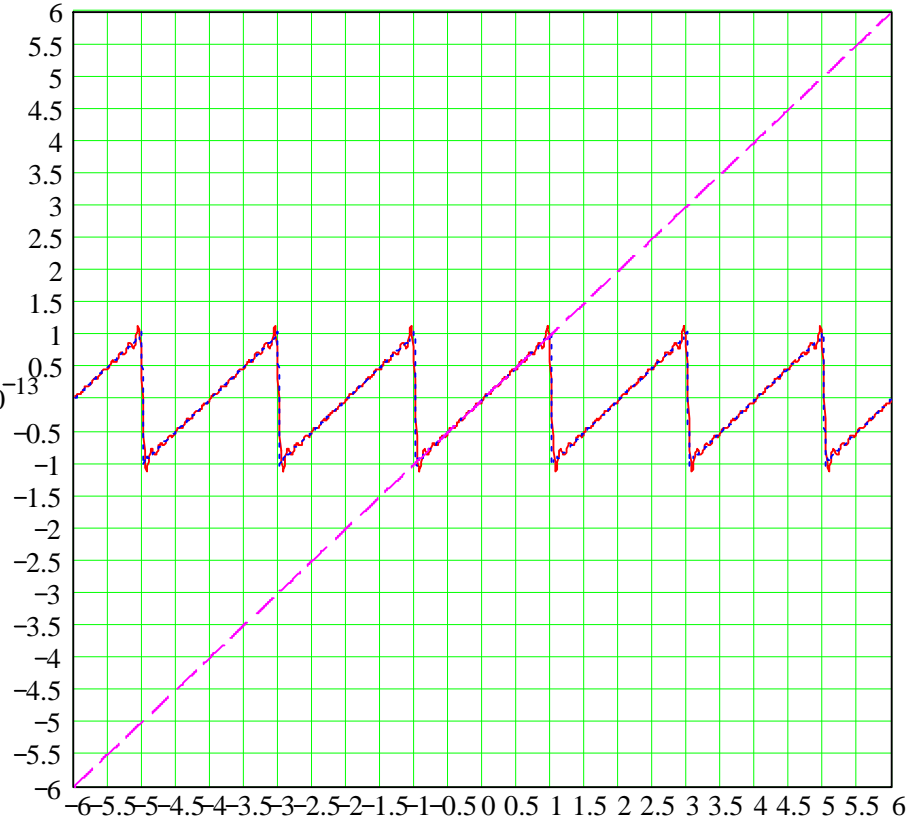
$$M := 6$$

$$x := -M \cdot L, -M \cdot L + 0.01 .. M \cdot L$$

$F(x, 15)$

$F(x, N_{max}) - 1.278976924368210 \cdot 10^{-13}$

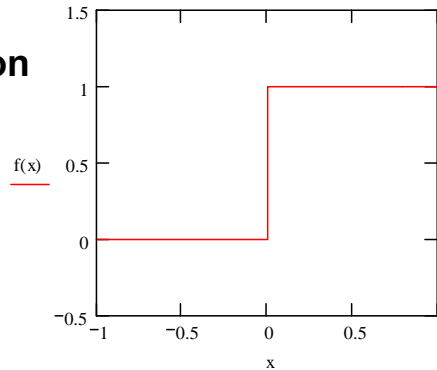
$x$



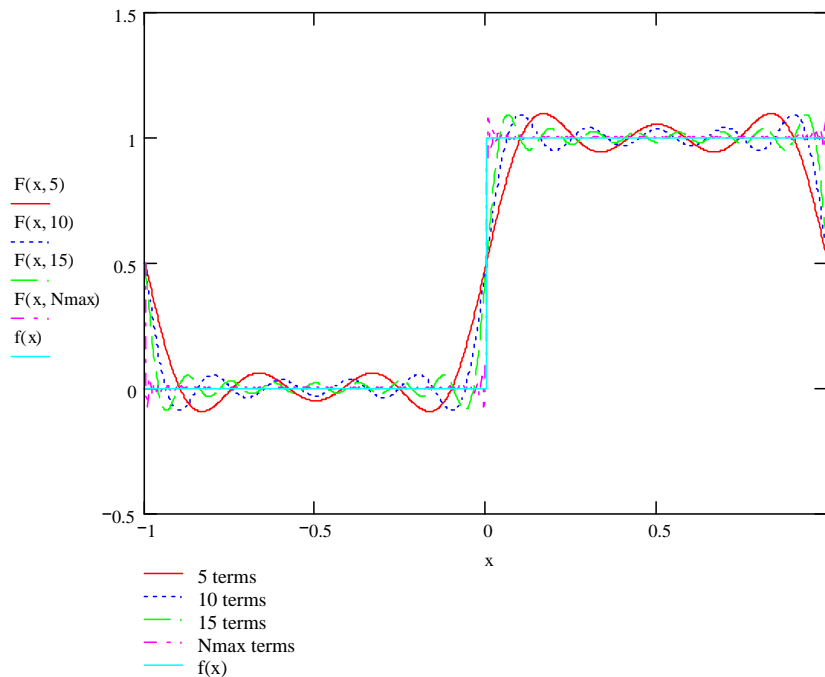
- 15 terms
- - - Nmax terms
- - -  $f(x)=x$

# Application of Series in Heat Transfer: transient heat conduction

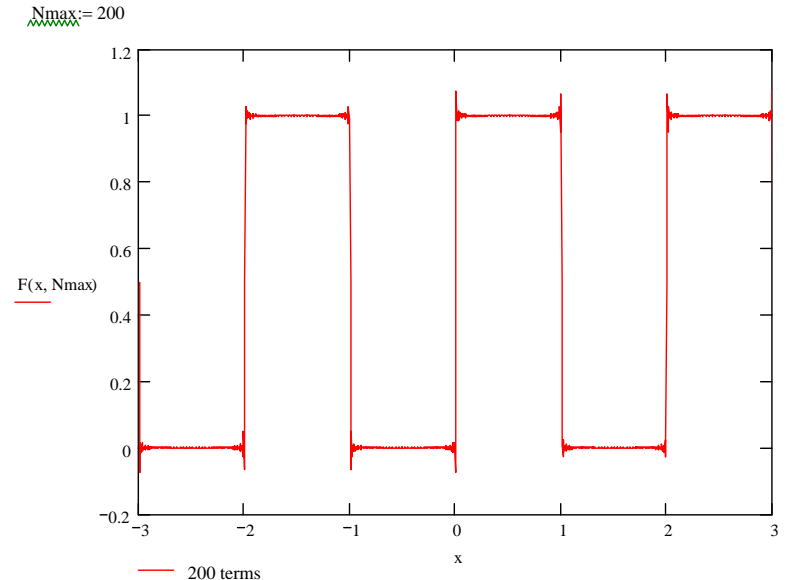
## Step function



$$F(x, N) := a_0 + \sum_{n=1}^N \left( a_n \cdot \cos\left(n \cdot \frac{\pi}{L} \cdot x\right) + b_n \sin\left(n \cdot \frac{\pi}{L} \cdot x\right) \right)$$

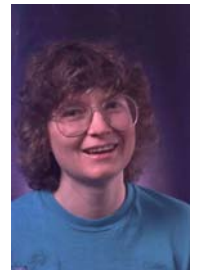


- Fourier series can “handle” even discontinuous functions



- modern version: I. Daubechie's wavelets handles jumps without wiggles

<http://www.pacm.princeton.edu/~ingrid/>





## Conclusions

- we studied two series:
  1. Taylor series - very powerful but can't model discontinuous functions.
    - applied to evaluation of error function (the probability integral)
  2. Fourier series - very general and can model to some degree discontinuous functions.
  
- Coming attractions: applications to heat transfer and finite difference methods in heat and fluid flow