Interactive Graphics Using Parametric Equations (Day 2)

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Computer Science

UCF EXCEL

Bezier Curves

Google “bezier curves``

- http://www.doc.ic.ac.uk/~dfg/AndysSplineTutorial/Beziers.html/

Interactive Graphics Curves

Uses:

-- Design of Fonts and other printer symbols

-- Consumer goods: shapes of cell phones, cars, etc.

Bezier Curves

Curve is specified by 2 equations:

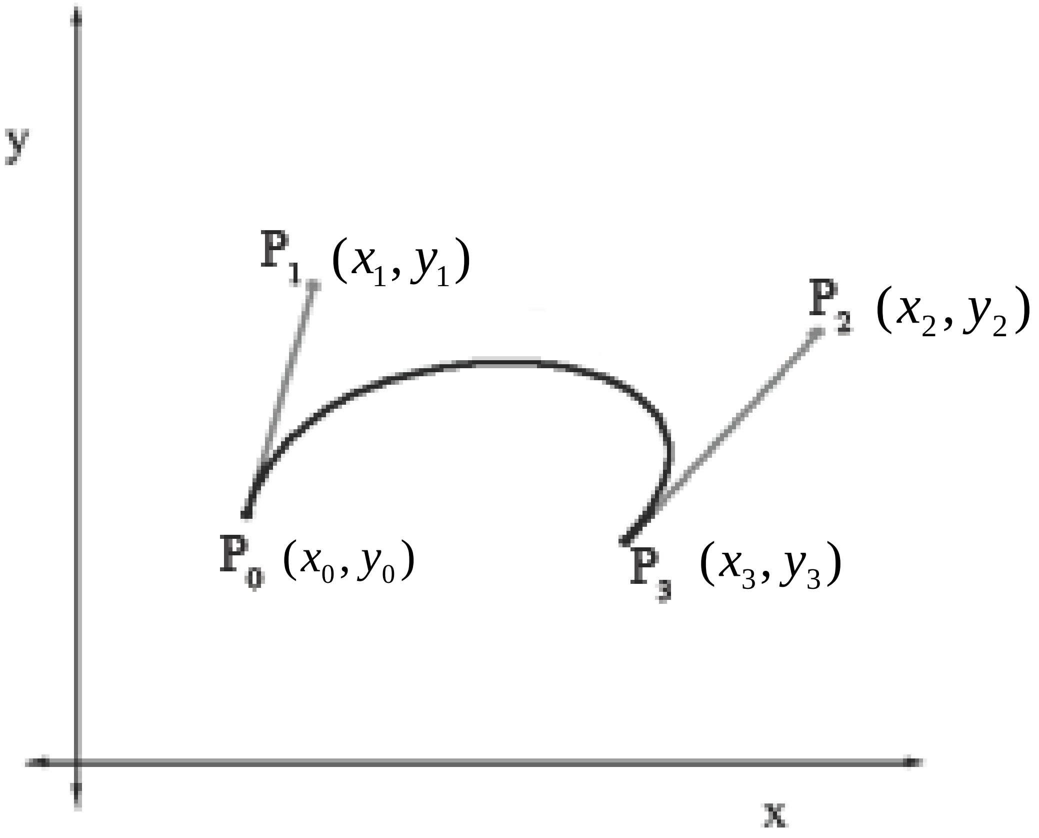
$$x = x_0(1-t)^3 + 3x_1t(1-t)^2 + 3x_2t^2(1-t) + x_3t^3$$

$$y = y_0(1-t)^3 + 3y_1t(1-t)^2 + 3y_2t^2(1-t) + y_3t^3$$

(x_0, y_0) and (x_3, y_3) are curve endpoints

(x_1, y_1) and (x_2, y_2) are guidepoints

A Bezier Curve



Bezier Curves

Mathematically, we verified that:

Slope of a handle is same as tangent at endpoint

i.e., the tangent at (x_0, y_0) has same slope as the segment joining (x_0, y_0) to (x_1, y_1)

Graphics Curve in General Form

Curve is specified by 2 equations, one is:

$$x = x_0(1-t)^3 + 3x_1t(1-t)^2 + 3x_2t^2(1-t) + x_3t^3$$

Which can be re-written as

$$\begin{aligned} x = \quad & x_0 \quad t^0 \quad (1-t)^3 \\ +3 \quad & x_1 \quad t^1 \quad (1-t)^2 \\ +3 \quad & x_2 \quad t^2 \quad (1-t)^1 \\ + \quad & x_3 \quad t^3 \quad (1-t)^0 \end{aligned}$$

General Form of Bezier Curve

This equation

$$x = x_0 \; t^0 \; (1-t)^3 + 3 \; x_1 \; t^1 \; (1-t)^2 + 3 \; x_2 \; t^2 \; (1-t)^1 + x_3 \; t^3 \; (1-t)^0$$

can be re-written as

$$x = \frac{3!}{0!(3-0)!} x_0 \; t^0 \; (1-t)^3 + \frac{3!}{1!(3-1)!} x_1 \; t^1 \; (1-t)^2 + \frac{3!}{2!(3-2)!} x_2 \; t^2 \; (1-t)^1 + \frac{3!}{3!(3-3)!} x_3 \; t^3 \; (1-t)^0$$

General Form of Bezier Curve

This form

$$x = \frac{3!}{0!(3-0)!} x_0 \quad t^0 \quad (1-t)^3$$

$$+ \frac{3!}{1!(3-1)!} x_1 \quad t^1 \quad (1-t)^2$$

$$+ \frac{3!}{2!(3-2)!} x_2 \quad t^2 \quad (1-t)^1$$

$$+ \frac{3!}{3!(3-3)!} x_3 \quad t^3 \quad (1-t)^0$$

can be re-written as

$$x = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} x_k \quad t^k \quad (1-t)^{(n-k)}$$

General Form of Bezier Curve

Hence, the equations are (using *n*=3, for 4 points)

$$x = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \quad x_k \quad t^k \quad (1-t)^{(n-k)}$$

$$y = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \quad y_k \quad t^k \quad (1-t)^{(n-k)}$$

Generic notation,

$$\vec{P} = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \quad \vec{p}_k \quad t^k \quad (1-t)^{(n-k)}$$

Or, $$\vec{P}(t) = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \quad \vec{p}_k \quad t^k \quad (1-t)^{(n-k)}$$

General Form of Bezier Curve

$$\vec{P}(t) = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \vec{p}_k \quad t^k \quad (1-t)^{(n-k)}$$

Can be re-written as,

$$\vec{P}(t) = \sum_{k=0}^{n} \vec{p}_k \frac{n!}{k!(n-k)!} \quad t^k \quad (1-t)^{(n-k)}$$

Or,

$$\vec{P}(t) = \sum_{k=0}^{n} \vec{p}_k \quad BEZ_{k,n}(t)$$

where the organizer can choose *n*, and the user then supplies *n+1* points.

General Form of Bezier Curve

$$\vec{P}(t) = \sum_{k=0}^{n} \vec{p}_k \; BEZ_{k,n}(t)$$

is,

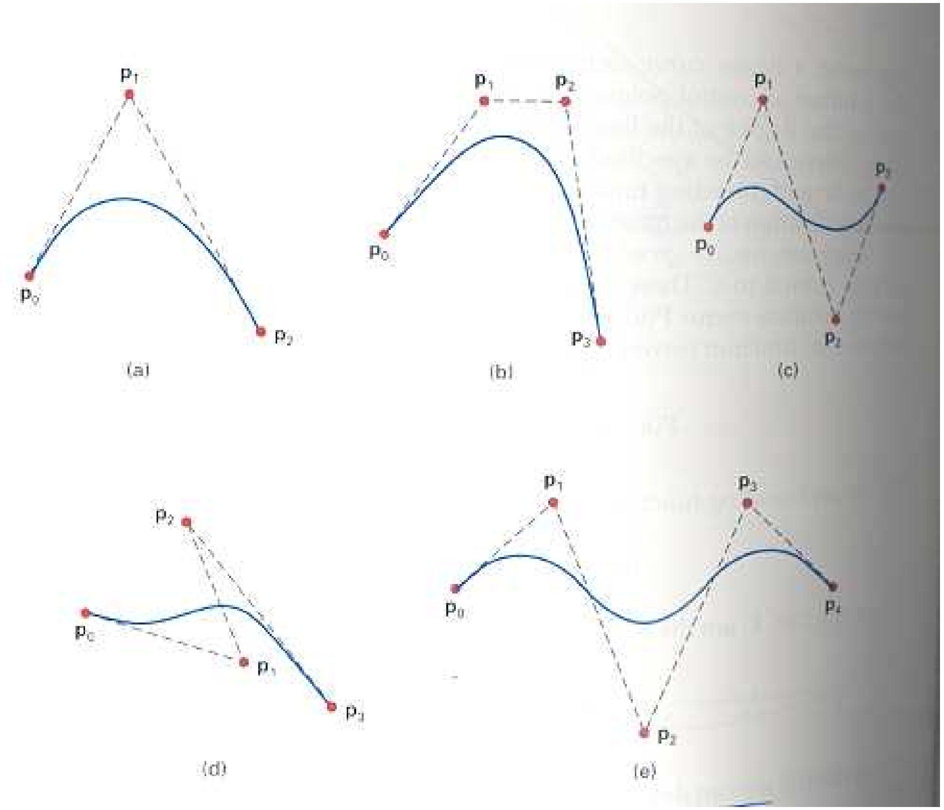
$$\vec{P}(t) = \sum_{k=0}^{n} \vec{p}_k \; \frac{n!}{k!(n-k)!} \; t^k \; (1-t)^{(n-k)}$$

Most common is $n=3$, some use $n=2$, and $n=4$.

Abandon notion of handles, and use notion of guidepoints.

General Form of Bezier Curve

Most common is *n=3*, some use *n=2*, and *n=4*.



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Bezier Curve in Matrix Notation

Consider the familiar case

$$x = x_0(1-t)^3 + 3x_1t(1-t)^2 + 3x_2t^2(1-t) + x_3t^3$$

It expands to

$$\begin{aligned} x = x_0 - 3x_0t + 3x_0t^2 - x_0t^3 \\ + 3x_1t - 6x_1t^2 + 3x_1t^3 \\ + 3x_2t^2 - 3x_2t^3 \\ + x_3t^3 \end{aligned}$$

Towards Matrix Notation

$$x = x_0 - 3x_0 t + 3x_0 t^2 - x_0 t^3$$
$$+ 3x_1 t - 6x_1 t^2 + 3x_1 t^3$$
$$+ 3x_2 t^2 - 3x_2 t^3$$
$$+ x_3 t^3$$

Gives

$$x = t^3(-1x_0 + 3x_1 - 3x_2 + x_3)$$
$$+ t^2(3x_0 - 6x_1 + 3x_2)$$
$$+ t(-3x_0 + 3x_1)$$
$$+ t^0(x_0)$$

Towards Matrix Notation

$$x = t^3(-1x_0 + 3x_1 - 3x_2 + x_3)$$
$$+ t^2(\ 3x_0 - 6x_1 + 3x_2)$$
$$+ t\ (-3x_0 + 3x_1)$$
$$+ t^0(x_0)$$

Can be written as

$$x = [t^3 \quad t^2 \quad t \quad 1]\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Matrix notation

Suppose we have 3 Apples + 5 Bananas = $10

6 Apples + 7 Bananas = $15

What is cost of Apple? cost of Banana?

Can write as $3A + 5B = 10$

$6A + 7B = 15$

In matrix notation, get

$$\begin{bmatrix} 3 & 5 \\ 6 & 7 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 10 \\ 15 \end{bmatrix}$$

Matrix notation

Using matrix notation,

then would "solve this Matrix system'' $\begin{bmatrix} 3 & 5 \\ 6 & 7 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 10 \\ 15 \end{bmatrix}$

for $\begin{bmatrix} A \\ B \end{bmatrix}$

by finding inverse matrix of $\begin{bmatrix} 3 & 5 \\ 6 & 7 \end{bmatrix}$

This is studied in class on

Matrix Algebra or Linear Algebra.

Matrix Notation

Matrices are also very useful in Computer Graphics for dealing with rotations and 3-dimensional projections

For now, we only care about the notation; i.e., that we can write in one form (English) or the other (matrix).

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| |

3 Apples + 5 Bananas = $10
6 Apples + 7 Bananas = $15

$$\begin{bmatrix} 3 & 5 \\ 6 & 7 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 10 \\ 15 \end{bmatrix}$$

Back To Bezier Curves

We had

$$x = t^3(-1x_0 + 3x_1 - 3x_2 + x_3)$$
$$+ t^2(\ 3x_0 - 6x_1 + 3x_2)$$
$$+ t\ (-3x_0 + 3x_1)$$
$$+ t^0(x_0)$$

Wrote as

$$x = [t^3 \quad t^2 \quad t \quad 1]\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Back To Bezier Curves

This

$$x = [t^3 \quad t^2 \quad t \quad 1] \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

has same form for y

Generic equation:

$$\vec{P}(t) = [t^3 \quad t^2 \quad t \quad 1] \, . \, M_{Bez} \, . \begin{bmatrix} \vec{p}_0 \\ \vec{p}_1 \\ \vec{p}_2 \\ \vec{p}_3 \end{bmatrix}$$

Recall the Polynomial Notation

For curve, $\vec{P}(t) = \sum_{k=0}^{n} \vec{p}_k \; BEZ_{k,n}(t)$

$$\vec{P}(t) = [t^3 \quad t^2 \quad t \quad 1] \,.\, M_{Bez} \,.\, \begin{bmatrix} \vec{p}_0 \\ \vec{p}_1 \\ \vec{p}_2 \\ \vec{p}_3 \end{bmatrix}$$

For surface, $\vec{P}(t,s) = \sum_{j=0}^{m} \sum_{k=0}^{n} \vec{p}_{j,k} \; BEZ_{j,m}(s) \; BEZ_{k,n}(t)$

Bezier Surfaces

We have $\vec{P}(t,s) = \sum_{j=0}^{m} \sum_{k=0}^{n} \vec{p}_{j,k} \; BEZ_{j,m}(s) \; BEZ_{k,n}(t)$

Or, for *m*=*n*=*3*

$$\vec{P}(t,s) = [t^3 \quad t^2 \quad t \quad 1] \,.\, M_{Bez} \,.\, \begin{bmatrix} \vec{p}_{00} & \vec{p}_{01} & \vec{p}_{02} & \vec{p}_{03} \\ \vec{p}_{10} & \vec{p}_{11} & \vec{p}_{12} & \vec{p}_{13} \\ \vec{p}_{20} & \vec{p}_{21} & \vec{p}_{22} & \vec{p}_{23} \\ \vec{p}_{30} & \vec{p}_{31} & \vec{p}_{32} & \vec{p}_{33} \end{bmatrix} \,.\, M_{Bez}^{T} \begin{bmatrix} s^3 \\ s^2 \\ s \\ 1 \end{bmatrix}$$

Bezier Surfaces

We need

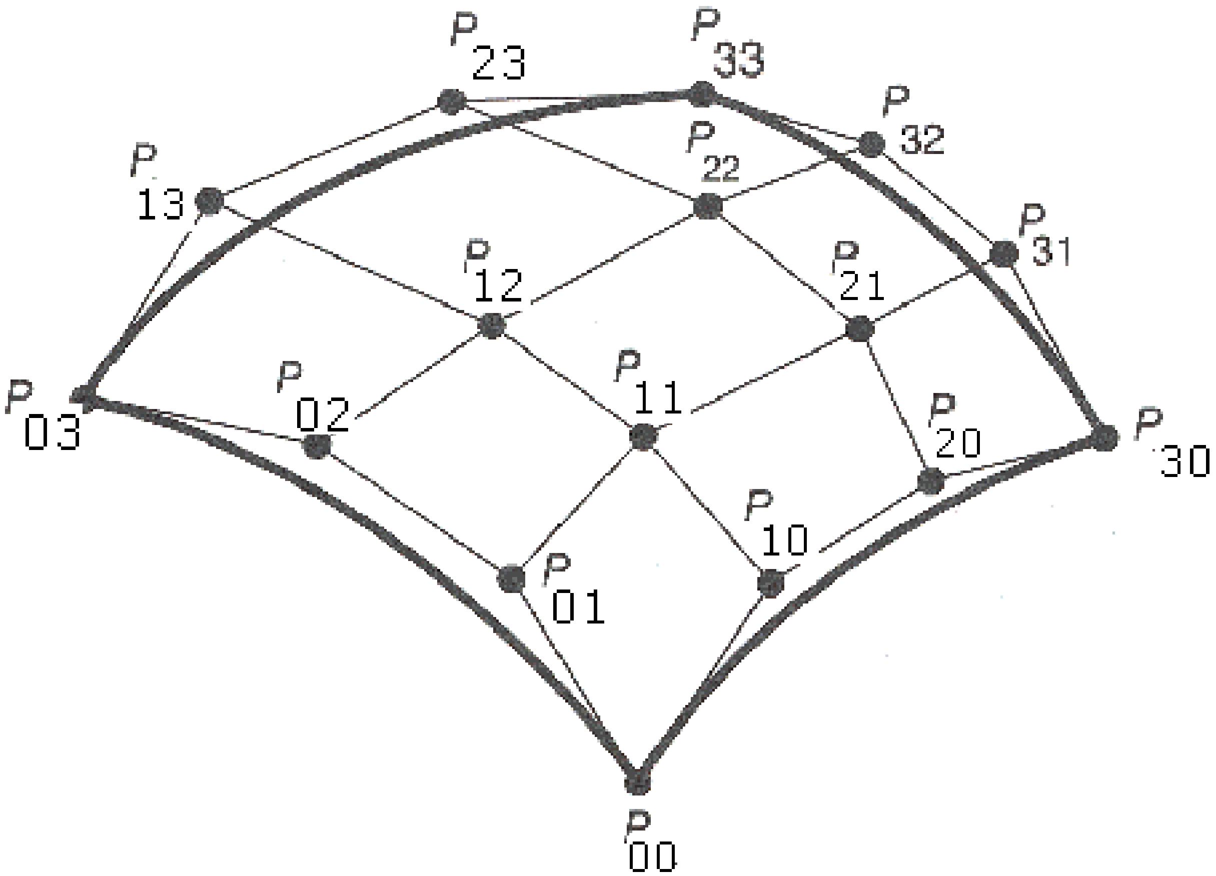

Figure adapted from Princeton Web site

Some Properties of Bezier Surfaces

1) Four corners are like Tent anchors, i.e., they are tied down to fixed points. (Interpolation)

2) Along any border, the surface behaves as a single Bezier curve.

3) Just as with single curve, the surface fits in the Convex Hull of the specified points

Some Properties of Bezier Surfaces

4) Because the four borders are explicit Bezier curves, they can be linked to neighboring patches, by making the common border the same Bezier curve, i.e., the same four control points.