Mathematical Modeling of the Population Growth of a Single Species

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UCF EXCEL Applications of Calculus
Improved Growth Model with Harvesting

The effect of Harvesting

1. Consider a species that is hunted or fished with a yearly quota specified. In this case, the differential equation model is modified as follows.

2. \[ x'(t) = rx(1 - \frac{x}{K}) - H \]

3. \( H \) is called the harvesting rate and is a constant. We are interested in values of \( H \) that do not lead to extinction. Note \( H \) tends to decrease the population as expected.
Equilibrium Solutions

1. Consider the case where $r = 1$, $K = 100$ and $H = 21$.
2. The DE is
   \[ x'(t) = x(1 - \frac{x}{100}) - 21 \]
3. When $H$ is not present, the equilibrium solutions are $x = 0$ and $100$.
4. With $H = 21$, we solve the quadratic:
   \[ x(1 - \frac{x}{100}) - 21 = 0 \]
Solution to DE

\[
\frac{dx}{dt} = x(1 - (x/100)) - 21, \ x(0) = 100
\]

1. Separate variables by dividing both sides by RHS.

\[
\frac{1}{x(1 - (x/100)) - 21} \frac{dx}{dt} = 1
\]

2. Integrate both sides

\[
\int \frac{1}{x(1 - (x/100)) - 21} \, dx = \int dt
\]
Solution to DE

3. Simplify the left hand side and factor:

\[ \int \frac{-100}{(x - 30)(x - 70)} \, dx = \int \, dt \]

4. Integrate using partial fractions on the left side.
Solution to DE

5. Algebraically solve for $x(t)$ and require that $x(0) = 100$ (this determines $C$)

$$x(t) = \frac{490 - 90e^{-2t/5}}{7 - 3e^{-2t/5}}$$

6. The large time behavior of the population will tell what happens eventually.
Harvesting H = 21
Increased Harvesting Rate

1. Consider the case where \( r = 1 \), \( K = 100 \) and \( H = 25 \).

2. The DE is

\[
x'(t) = x(1 - \frac{x}{100}) - 25
\]

3. With \( H = 25 \), we solve the quadratic:

\[
x(1 - \frac{x}{100}) - 25 = 0
\]

to find the equilibrium solutions.
Solution to DE

4. Require that $x(0) = 100$ which gives $C = 2$. The solution is then

$$x(t) = \frac{100}{t + 2} + 50$$

5. The large time behavior of the population will tell what happens eventually.
Increased Harvesting Rate

1. Consider the case where \( r = 1, K = 100 \) and \( H = 29 \).
2. The DE is
\[
x'(t) = x(1 - \frac{x}{100}) - 29
\]
3. With \( H = 29 \), we solve the quadratic:
\[
x(1 - \frac{x}{100}) - 29 = 0
\]
Solution to DE

\[ \frac{dx}{dt} = x(1 - (x/100)) - 29, \quad x(0) = 100 \]

1. Separate variables by dividing both sides by RHS. Simplify the expression to get

\[ \int \frac{-100}{(x^2 - 100x + 2900)} dx = \int dt \]

2. To integrate write \((x^2 - 100x + 2900) = (x - 50)^2 + 400.\)
Solution to DE

3. Introduce the change of variable
   \[ y = x - 50 \]

   \[ \int \frac{-100}{y^2 + 20^2} \, dy = \int \, dt \]

4. Integration gives an arctan function.
   \[ -5 \arctan\left(\frac{x - 50}{20}\right) = t + C \]
Solution to DE
5. Require that \( x(0) = 100 \) to find \( C \).

\[
x(0) = 100 = 50 + 20 \tan C
\]

6. Solve for \( C \) to find

\[
C = \tan^{-1}\left(\frac{5}{2}\right) \quad x(t) = 50 + 20 \tan(C - (t/5)).
\]

7. This solution corresponds to extinction.
Harvesting $H = 29$
Concluding Remarks
When $H < 25$, the harvesting rate still allows the population to tend to a constant equilibrium value.
When $H = 25$, this is the transition point between having a sustainable population and one that goes extinct from say, overfishing.
When $H > 25$, the population will always go extinct.
A plot of $x' = x(1-(x/100)-H)$ vs $x$ gives insight.
Improved Growth Model with Harvesting

Red $H = 21$  Green $H = 25$  Yellow $H = 29$
Other Problems

1. Take $K = \text{function of } t \text{ and/or } x$.
2. Age structure
3. Population depends on past history
4. Predator – prey
5. Two species competing for the same resources
6. Two species cooperating
Courses

1. MAP2302 Differential Equations
2. MAP4103 Mathematical Modeling