Mathematical Modeling of the Population Growth of a Single Species

by

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Improved Growth Model with Harvesting

- **The effect of Harvesting**

  1. Consider a species that is hunted or fished with a yearly quota specified. In this case, the differential equation model is modified as follows

  2. DE $x'(t) = rx(1 - (x/K)) - H$

  3. H is called the harvesting rate and is a constant. We are interested in values of H that do not lead to extinction. Note H tends to decrease the population as expected.

Equilibrium Solutions

1. Consider the case where $r = 1$, $K = 100$ and $H = 21$.
2. The DE is

$$x'(t) = x(1 – (x/100)) - 21$$

3. When H is not present, the equibrium solutions are $x = 0$ and 100.
4. With $H = 21$, we solve the quadratic:

$$x(1 – (x/100)) – 21 = 0$$

Harvesting H = 21

Solution to DE

$$\frac{dx}{dt} = x(1 - (x/100)) - 21,\ x(0) = 100$$

1. Separate variables by dividing both sides by RHS.

$$\frac{1}{x(1 - (x/100)) - 21}\frac{dx}{dt} = 1$$

2. Integrate both sides

$$\int \frac{1}{x(1 - (x/100)) - 21}\,dx = \int dt$$

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Harvesting H = 21

- **Solution to DE**

3. Simplify the left hand side and factor:

$$\int \frac{-100}{(x-30)(x-70)}dx = \int dt$$

4. Integrate using partial fractions on the left side.

Harvesting H = 21

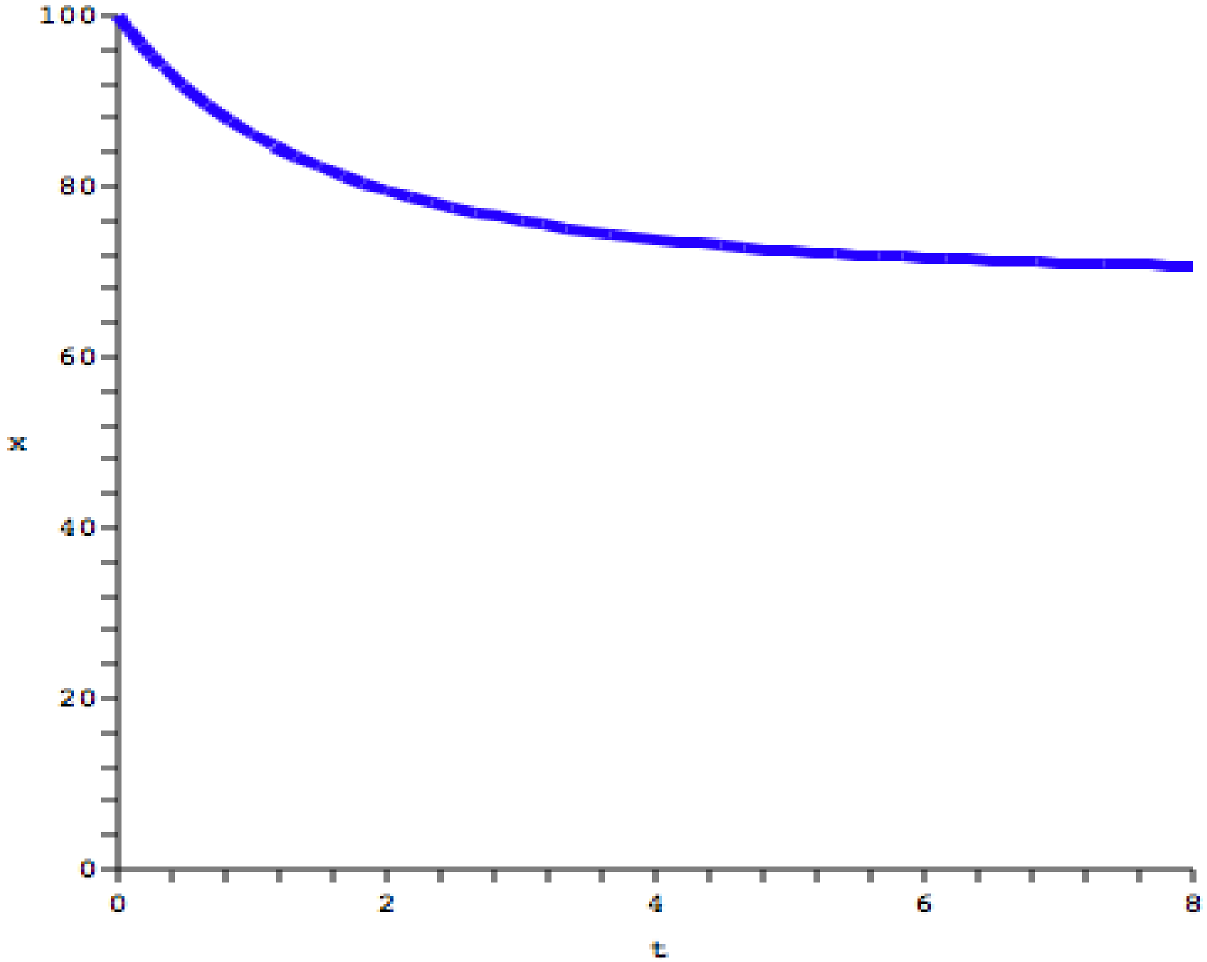
Solution to DE

5. Algebraically solve for x(t) and require that x(0) = 100 (this determines C)

$$x(t) = \frac{490 - 90e^{-2t/5}}{7 - 3e^{-2t/5}}$$

6. The large time behavior of the population will tell what happens eventually.

Harvesting H = 21



Increased Harvesting Rate

1. Consider the case where $r = 1$, $K = 100$ and $H = 25$.
2. The DE is

$$x'(t) = x(1 - (x/100)) - 25$$

3. With $H = 25$, we solve the quadratic:

$$x(1 - (x/100)) - 25 = 0$$

to find the equilibrium solutions.

Harvesting H = 25

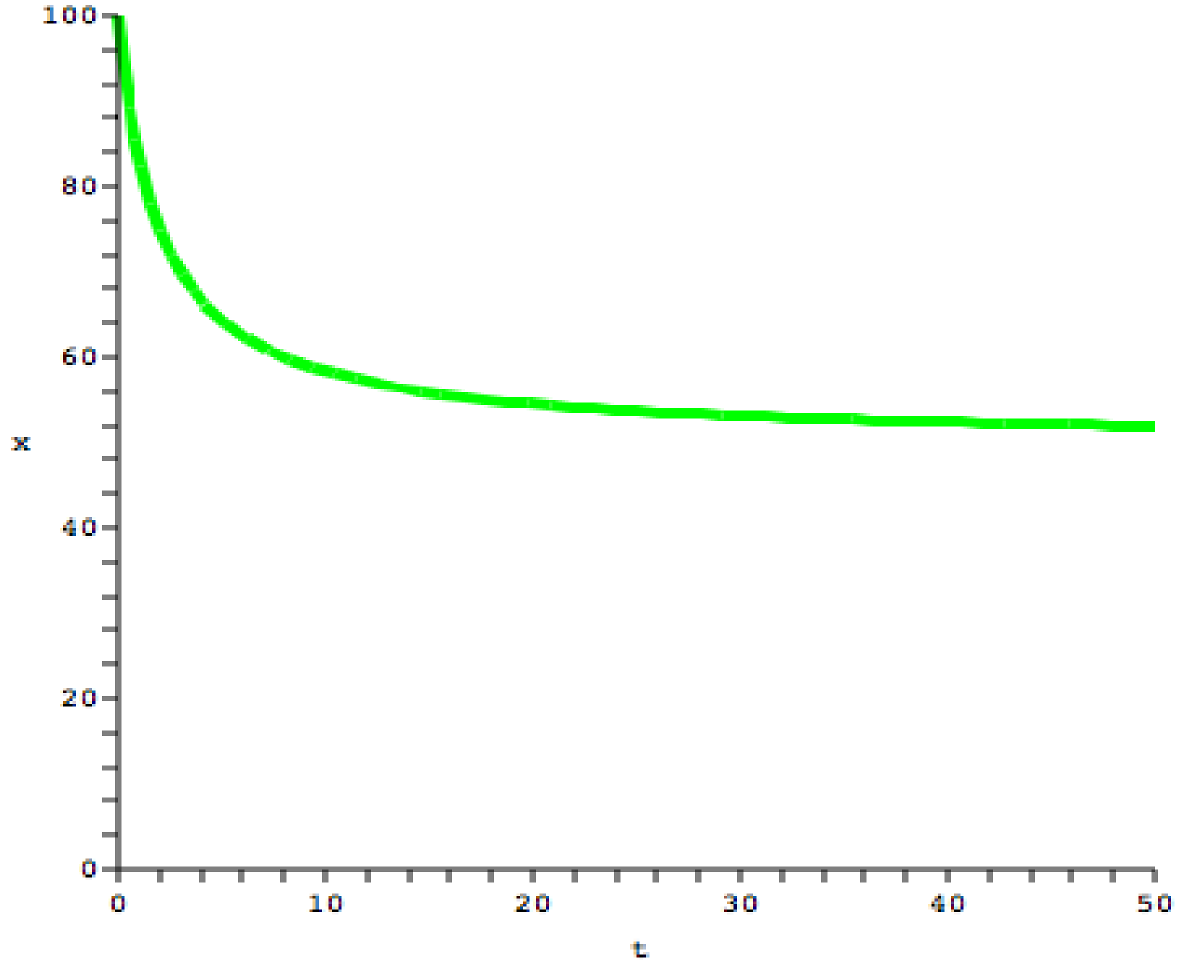
- **Solution to DE**

  4. Require that x(0) = 100 which gives C = 2. The solution is then

$$x(t) = \frac{100}{t+2} + 50$$

  5. The large time behavior of the population will tell what happens eventually.

Harvesting H = 25



Harvesting H = 29

- **Increased Harvesting Rate**

1. Consider the case where r = 1, K = 100 and H = 29.
2. The DE is

$$x'(t) = x(1 – (x/100)) - 29$$

3. With H = 29, we solve the quadratic:

$$x(1 – (x/100)) – 29 = 0$$

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Harvesting H = 29

- **Solution to DE**

$$\frac{dx}{dt} = x(1 - (x/100)) - 29,\ x(0) = 100$$

1. Separate variables by dividing both sides by RHS. Simplify the expression to get

$$\int \frac{-100}{(x^2 - 100x + 2900)} dx = \int dt$$

2. To integrate write $(x^2 – 100x +2900) = (x – 50)^2 + 400$.

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Solution to DE

3. Introduce the change of variable
y = x – 50

$$\int \frac{-100}{y^2 + 20^2} dy = \int dt$$

4. Integration gives an arctan function.

$$-5 \arctan((x - 50)/20) = t + C$$

Harvesting H = 29

- **Solution to DE**

5. Require that x(0) = 100 to find C.

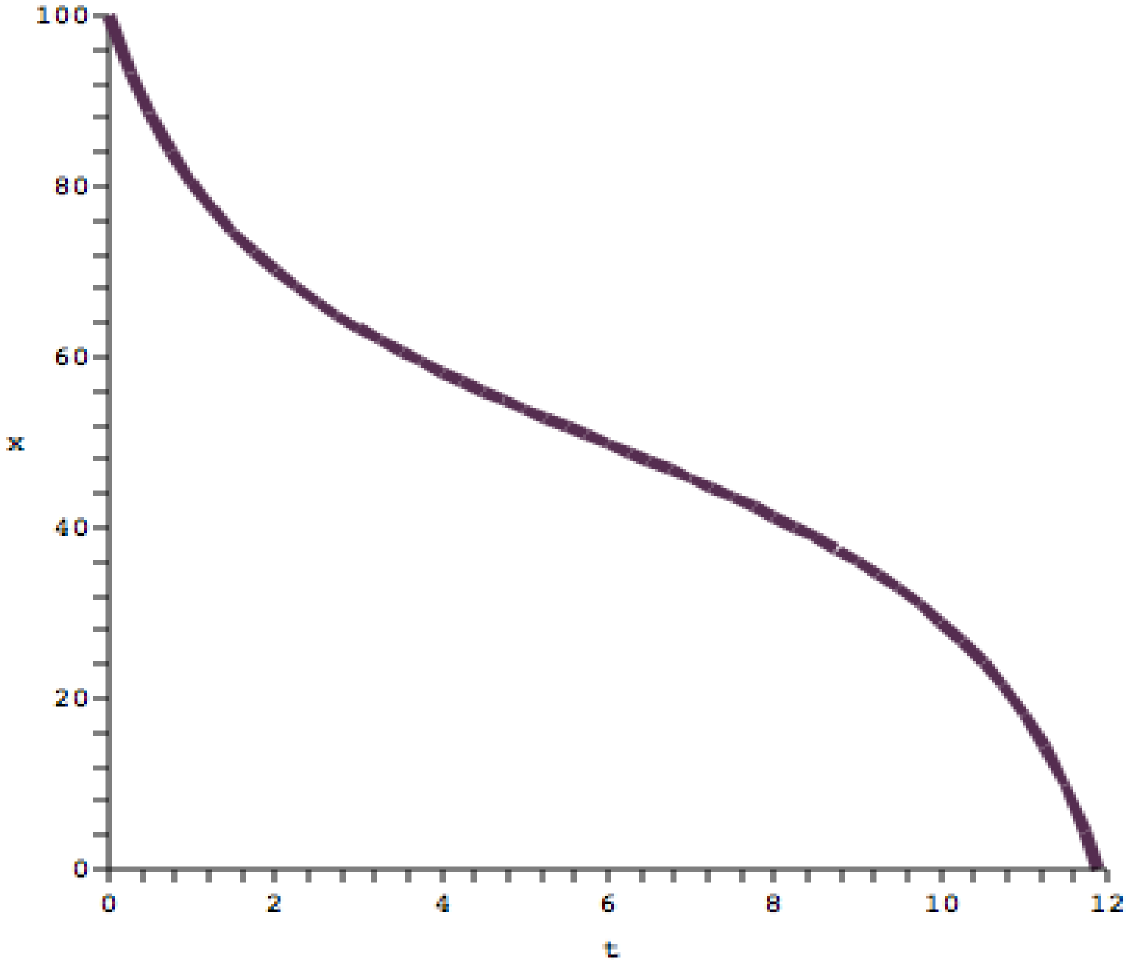
$$x(0) = 100 = 50 + 20\tan C$$

6. Solve for C to find

$$C = \tan^{-1}(5/2) \qquad x(t) = 50 + 20\tan(C - (t/5)).$$

7. This solution corresponds to extinction.

Harvesting H = 29



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Improved Growth Model with Harvesting

- **Concluding Remarks**

  When H < 25, the harvesting rate still allows the population to tend to a constant equilibrium value.

  When H = 25, this is the transistion point between having a sustainable population and one that goes extinct from say, over fishing.

  When H > 25, the population will always go extinct.

  A plot of x' (= x(1-(x/100)-H) vs x gives insight.

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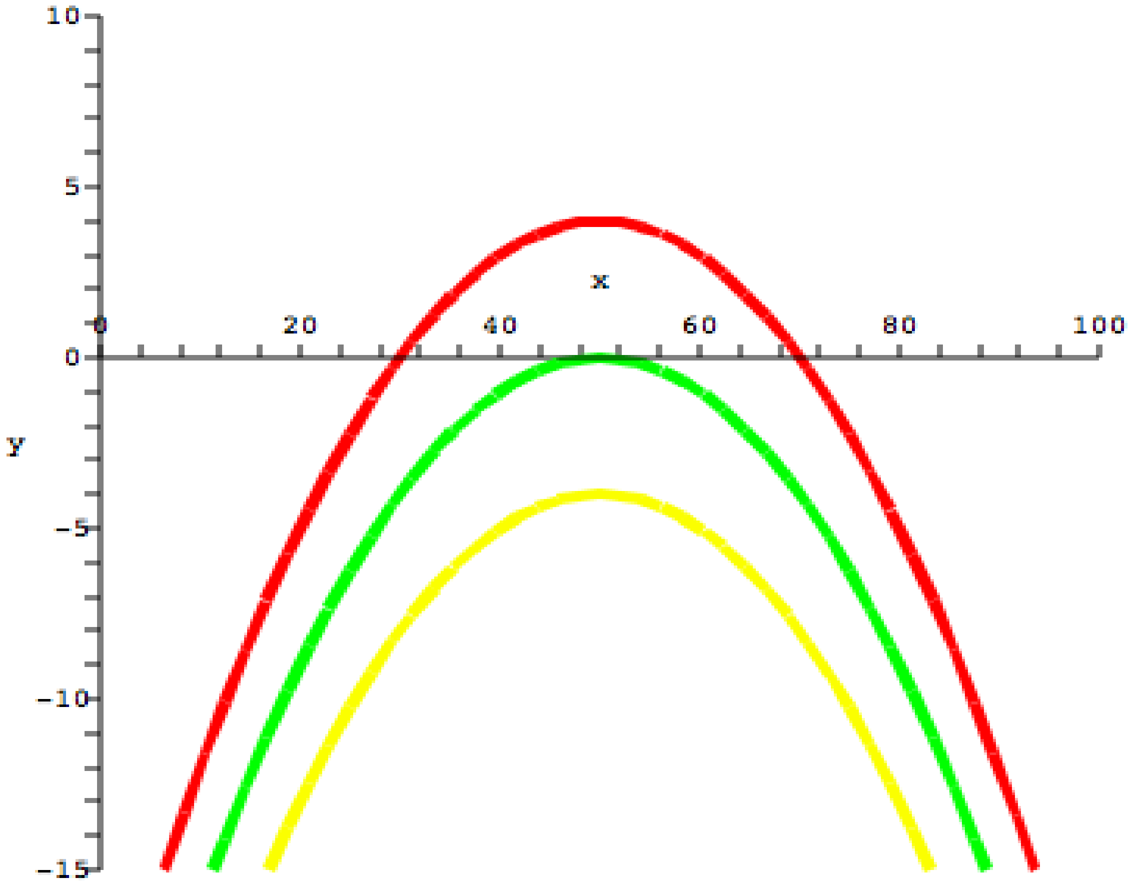
Improved Growth Model with Harvesting

Red H = 21 Green H = 25 Yellow H = 29



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What's Next?

Other Problems

1. Take K = function of t and/or x.
2. Age structure
3. Population depends of past history
4. Predator – prey
5. Two species competing for the same resourses
6. Two species cooperating

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Courses

1. MAP2302 Differential Equations
2. MAP4103 Mathematical Modeling