Mathematical Modeling of the Population Growth of a Single Species

by

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Calculus Topic: Integration of Rational Functions

Find the anti-derivative

$$\int \frac{1}{(t+4)(t-1)} dt$$

- Partial fraction expansion

$$\frac{1}{(t+4)(t-1)} = \frac{A}{t+4} + \frac{B}{t-1}.$$

- Coefficient evaluation

$$\frac{1}{(t+4)(t-1)} = \frac{A}{t+4} + \frac{B}{t-1} = \frac{A(t-1)+B(t+4)}{(t+4)(t-1)}.$$

$$\implies 1 = A(t-1) + B(t+4)$$

Setting $t = 1$ gives $1 = 5B$ and setting $t = -4$ gives $1 = -5A$.

- Simpler integral

$$\int \left(\frac{-\frac{1}{5}}{t+4} + \frac{\frac{1}{5}}{t-1} \right) dt = -\frac{1}{5} \int \frac{dt}{t+4} + \frac{1}{5} \int \frac{dt}{t-1}$$

Anti-derivatives is natural logs

$$\int \frac{1}{(t+4)(t-1)} dt = \frac{1}{5}(\ln|t-1| - \ln|t+4|) + C = \frac{1}{5} \ln\left|\frac{t-1}{t+4}\right| + C$$

Goals:

1. Understand what a differential equation is
2. Know how to solve a simple differential equation
3. Understand how a simple differential equations can be used to model population growth in a single species

Modeling of the Population Growth of a Single Species

- **Background:**

  1. In biology, mathematical models (integral or differential or difference equations) have been used for many years to describe the population of a particular species, whether it be human, bacteria or an endangered species.

  2. A mathematical model for the population model should find qualitative and quantitative behavior.

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Modeling of the Population Growth of a Single Species

- **Assumptions:**

  1. Deterministic - the future determined by the present conditions.
  2. Population of a given species can be represented by a continuous, differentiable function of time $x(t)$. This is a good approximation if the population is large.
  3. Population is in isolation - no predators or competitors.

What is a Differential Equation (DE)?

1. An equation relating an unknown function to one of more of its derivatives.
2. The order of a DE is determined by the highest derivative present.
3. $x'(t)=4x(t)+t$ is a first order differential equation.
4. Newton's second law: $mx''(t)=F(x(t),x'(t),t)$ where m is the mass, F is the applied force. This is a second order DE.

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- **What is a solution to a DE?**

  1. A function that on on substitution satisfies the equation. In general the solution to a differential equation is an infinite family of functions.

  2. To choose one of these functions to be the solution, require that the solution pass through a particular point. That is x(a)=b. This is called an initial condition as it tells the state of the system initially. Equation plus initial condition is called an IVP (initial value problem).

- **Solution Methods for DE**

  1. Analytical techniques - integration, transform methods, symmetry methods

  2. Numerical methods - finite difference. This is very important as many DE's can not be easily solved using analytical techniques.

  For this presentation we will make use of integration to solve a DE.

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- **Simplest Growth Model**

  1. Let x(t) be the population of an isolated species at any time t. x(t) > 0.
  2. Only consider the effects of births and deaths.
  3. The rate of change of x(t) with respect to t is equal to the birth rate – death rate.
  4. Assume that the birth term is proportional to x which is reasonable as we expect the number of births to increase with x. Similarly assume the death term is also proportional to x.

Simple Growth Model for Population Growth

- **Simplest Growth Model**

1. $x'(t)$ = birth rate – death rate

2. Assuming rates proportional to current population, then

$$x'(t) = bx - \mu x$$

where b is the per capita birth rate and μ is the per capita death rate.

3. It is natural to define a net growth rate constant $r = b - \mu$.

4. IVP $x'(t) = rx$ $x(0)$ specified.

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Simplest Growth Model

Qualitative features of $x'(t) = rx$

1. If $r > 0$ and since $x(t) > 0$ ($x = 0$ mean extinction of the population), then $x'(t) > 0$.
2. This means $x(t)$ is an increasing function of time. This makes sense since the birth rate is larger than the death rate.

- **Solving Separable DE's $x'(t) = A(x)B(t)$**

1. A separable equation is one where the two variables can be separated as shown.
2. Divide both sides of the DE by $A(x)$.
3. Integrate both sides of DE wrt t. Don't forget the integration constant C.
4. For IVP, determine C by requiring $x(a)=b$.

In applications, one should interpret the solution.

Solving the simple model

- **Solving x'(t) = rx(t)**

  1. Separate the variables by dividing both sides by x(t).

$$\frac{1}{x}\frac{dx}{dt} = r$$

  2. Integrate both sides of DE wrt t.

$$\int \frac{1}{x}\frac{dx}{dt}dt = \int r dt$$

  3. LHS: Change variable dx = (dx/dt) dt

Solving the simple model

- **Solving $x'(t) = rx(t)$**

4. Integrate both sides of:

$$\int \frac{1}{x}dx = \int r dt$$

5. This is a simple example of partial fractions.

$$\ln |x(t)| = rt + C$$

Note: Drop absolute values as $x(t) > 0$.

Solving the simple model

- **Solving $x'(t) = rx(t)$**

6. Take the exponential of both sides:

$$x(t) = e^{rt + C} = e^{rt} e^{C}$$

7. Since C is a constant, so is e^{C}

Re-define this as C, then $x(t) = Ce^{rt}$

8. Find C by satisfying the initial condition.
Say $x(0) = 1000$, then

$$x(0) = 1000 = Ce^{r0} = C$$

9. Solution $x(t) = 1000e^{rt}$

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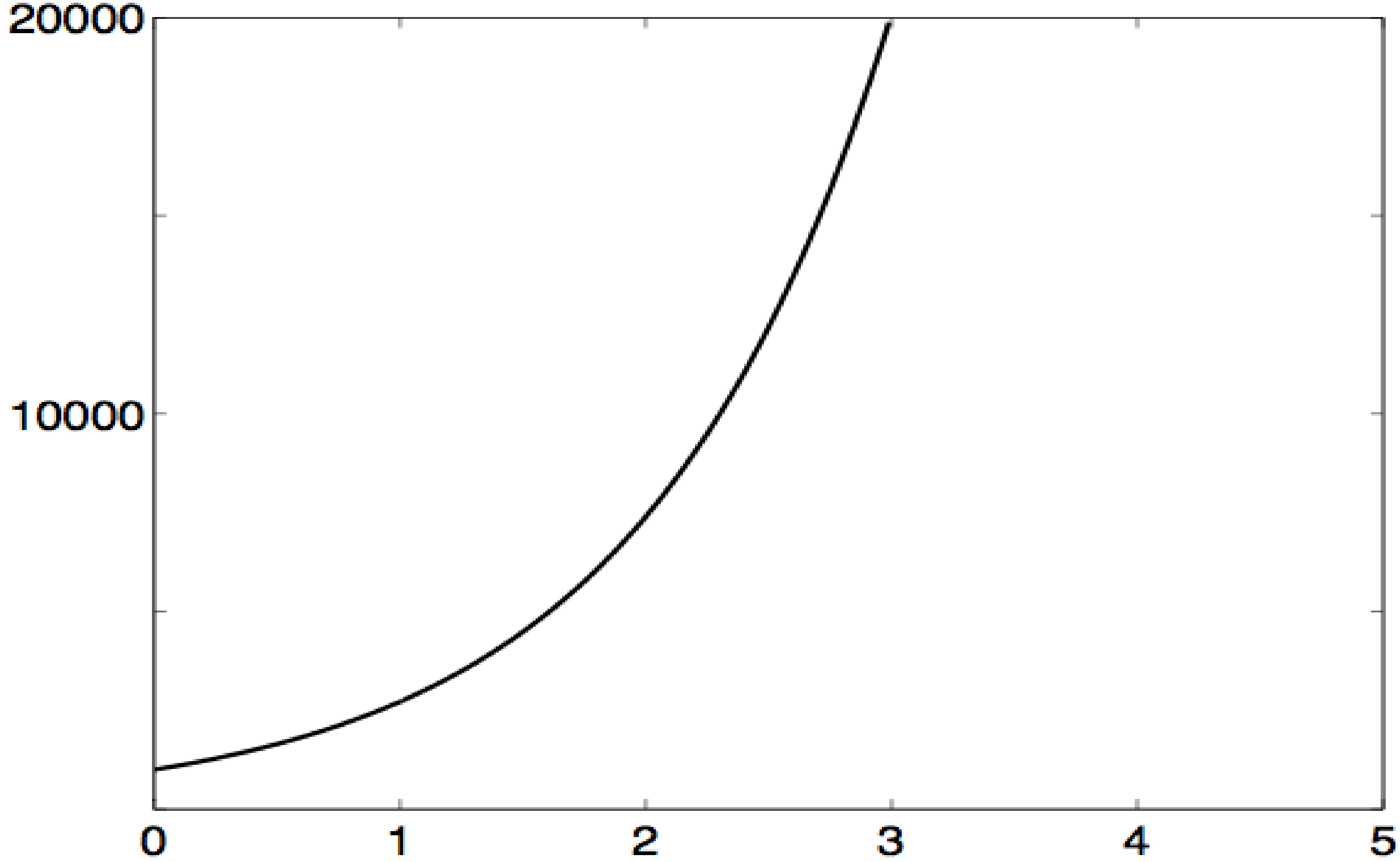
Simple Growth Model for Population Growth

Solution for r > 0.



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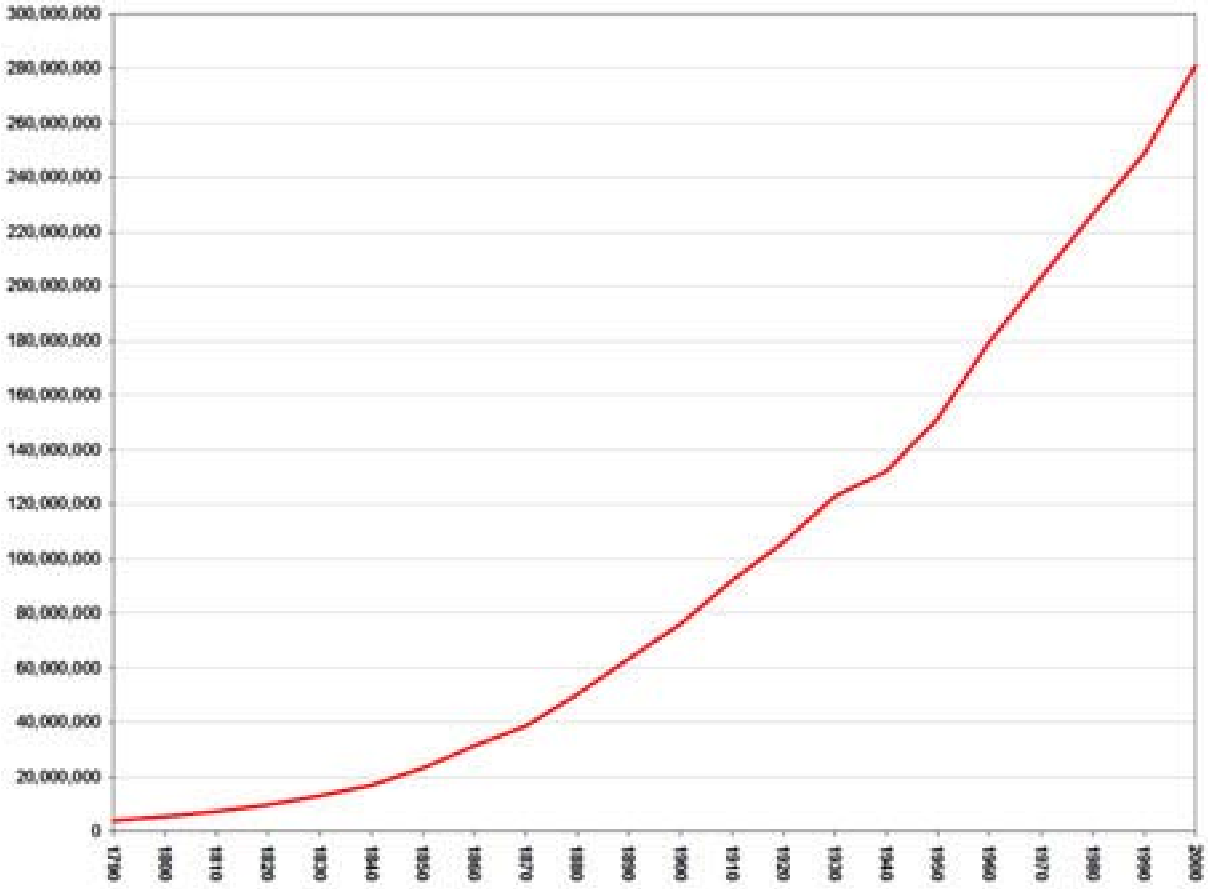
US Population 1750 to 2000



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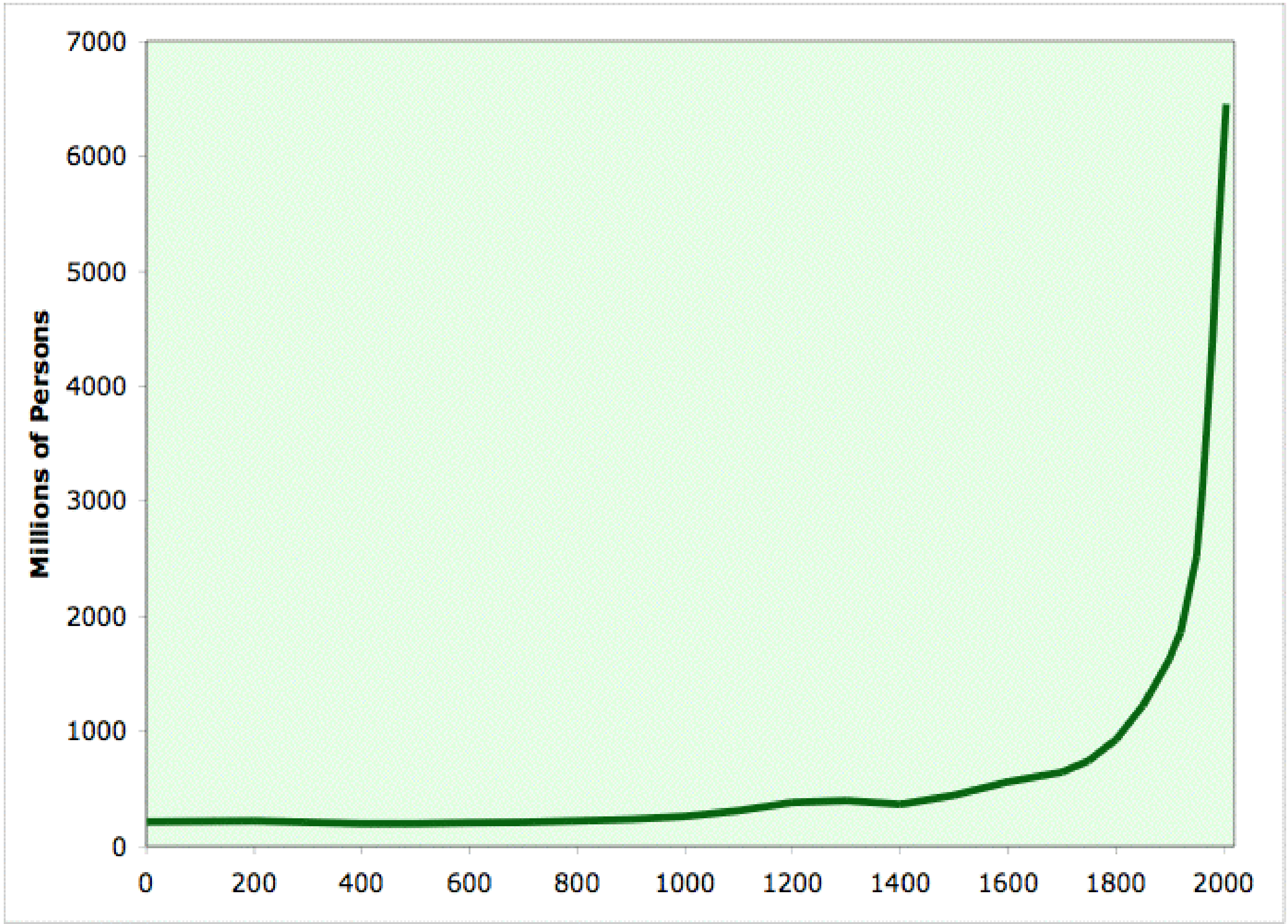
World Population to 2000



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- **Interpretation of solution**

1. Since r > 0, this is a growing exponential and the large time behavior is given by

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} 1000e^{rt} = \infty.$$

2. This means a population explosion with no way to stop the growth. But we know that
other factors will come into play. We have not included the fact that there are limited resources (food, land etc) into our mathematical model.

- **Improved Growth Model (Verhulst)**

  1. The simple model we have been examining can be improved by building into our differential equation a way to prevent the runaway growth when $r > 0$. We want the death rate to increase when x starts to become too large. This is reasonable as we start to use up our resources when the population is too large.
  2. DE $x'(t) = rx(1 – (x/K)) = rx - (rx^2 /K)$
  3. r is the net growth rate and K is the carrying capacity. Both are positive.

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Improved Growth Model for Population Growth

- **Improved Growth Model (Verhulst)**

  4. Note now that when x is large that the second term dominates the first (why?). So the derivative $x'(t)$ is now negative and so the population will decrease. Runaway growth is not possible.

  5. Note now that when x is small, the first term dominates the second. So the derivative $x'(t)$ is now positive and so the population will increase.

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Equilibrium Solutions

1. Equilibrium or steady state solutions are constant solutions to a DE.

2. Equilibrium solutions are found by setting the derivative equal to zero and solving the resulting equation.

3. For $x'(t) = f(x)$ then the equibrium solutions satisfy $f(x) = 0$.

4. For our problem $rx(1 – (x/K)) = 0$, so they are $x = 0$ and $x = K$.

Solution to DE

1. To simplify the integration, take K = 1. You may take that to mean that the population x(t) represents the % of the carrying capacity with x = 1 meaning 100%.

$$x(t) = rx(1 - x)$$

2. The equilibrium solutions are x = 0 and 1.

3. Assume x(0) = 2 is the initial condition which means the starting population is double the carrying capacity.

Solution to DE

$$\frac{dx}{dt} = rx(1-x).$$

1. Separate variables by dividing both sides by x(1 – x).

$$\frac{1}{x(1-x)}\frac{dx}{dt} = r$$

2. Integrate both sides with respect to t.

$$\int \frac{1}{x(1-x)}\frac{dx}{dt}dt = \int r dt$$

Solution to DE

3. On LHS, change variable to x by
dx = (dx/dt) dt.

$$\int \frac{1}{x(1-x)}dx = \int r dt$$

4. Integrate using partial fractions on the left side.

$$\frac{1}{x(1-x)} = \frac{-1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}$$

A = 1 and B = -1.

- **Solution to DE**

5. Integrate the partial fraction expansion.

$$\int\left(\frac{1}{x}-\frac{1}{x-1}\right)dt=\ln|x|-\ln|x-1|=rt+C.$$

6. Combining the logs

$$\ln\left|\frac{x}{x-1}\right|=rt+C$$

7. Inversion

$$\frac{x}{x-1}=e^{C}e^{rt}=C_0e^{rt}.$$

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Solution to DE

8. Suppose x(0)=2. Set t = 0 and solve for C.

$$\frac{2}{2-1} = C_0 e^0 \qquad \Rightarrow \qquad C_0 = 2.$$

9. Solve (7) algebraically for x(t).

$$x(t) = \frac{2e^{rt}}{2e^{rt}-1} = \frac{2}{2-e^{-rt}}$$

This gives the population at any time t.

- **Interpretation of the solution**

  Assuming r > 0, the long time behavior is

$$\lim_{t \to \infty} x(t) = \lim_{t \to \infty} \frac{2}{2 - e^{-rt}} = 1.$$

  This means a population tends toward the equilibrium population x(t) = 1 which is the carrying capacity for our population. This is typical behavior in many biological problems.

- **Interpretation of the solution**

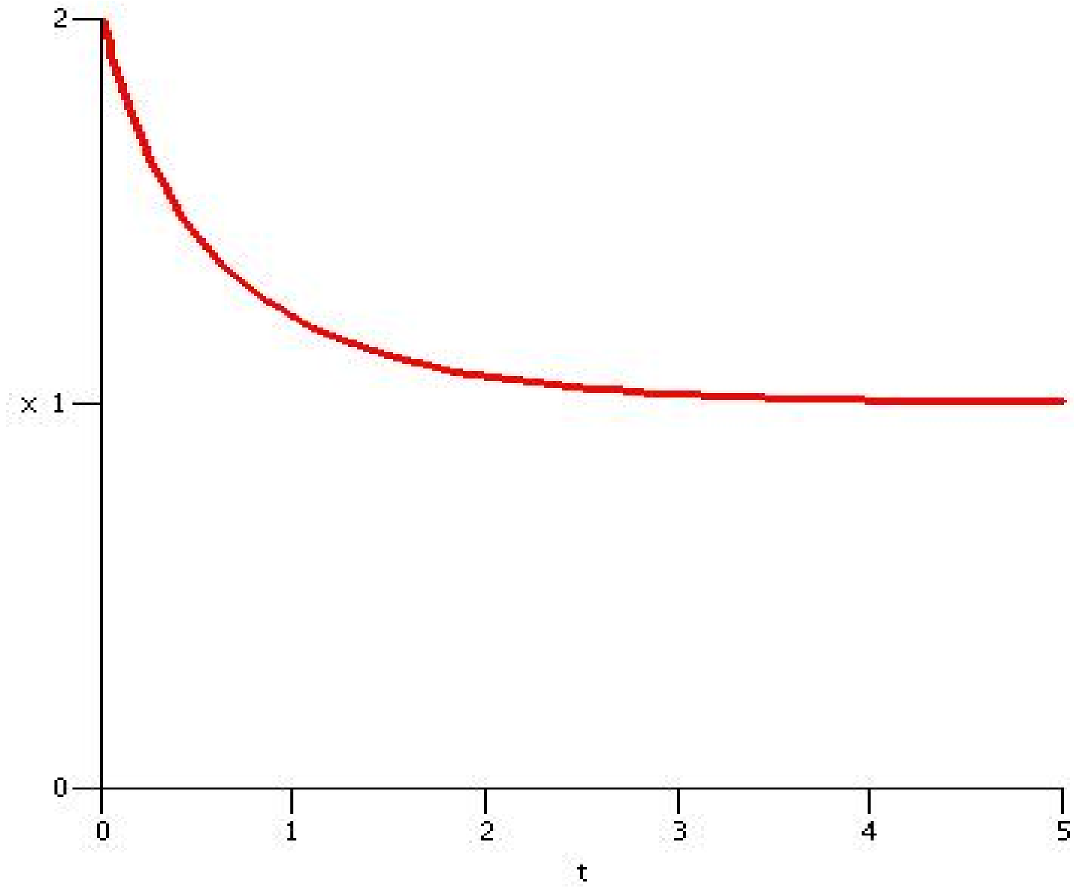
  In fact for any initial condition, the long time behavior is the same since:

$$x(t) = \frac{C_0 e^{rt}}{C_0 e^{rt} + (1 - C_0)} = \frac{C_0}{C_0 + (1 - C_0)e^{-rt}}$$

which tends toward the equilibrium population x(t) = 1.

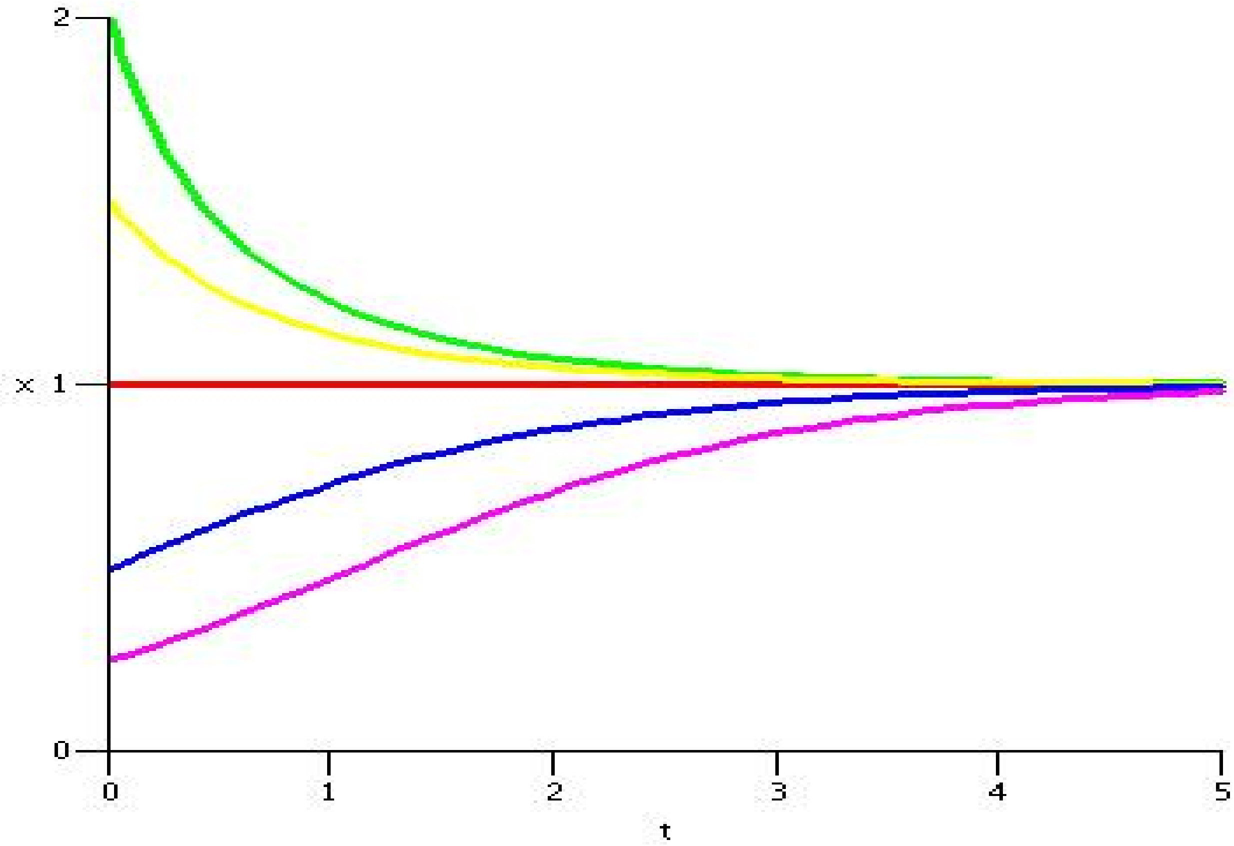
Improved Growth Model for Population Growth

Solution for r > 0 and x(0) = 2.



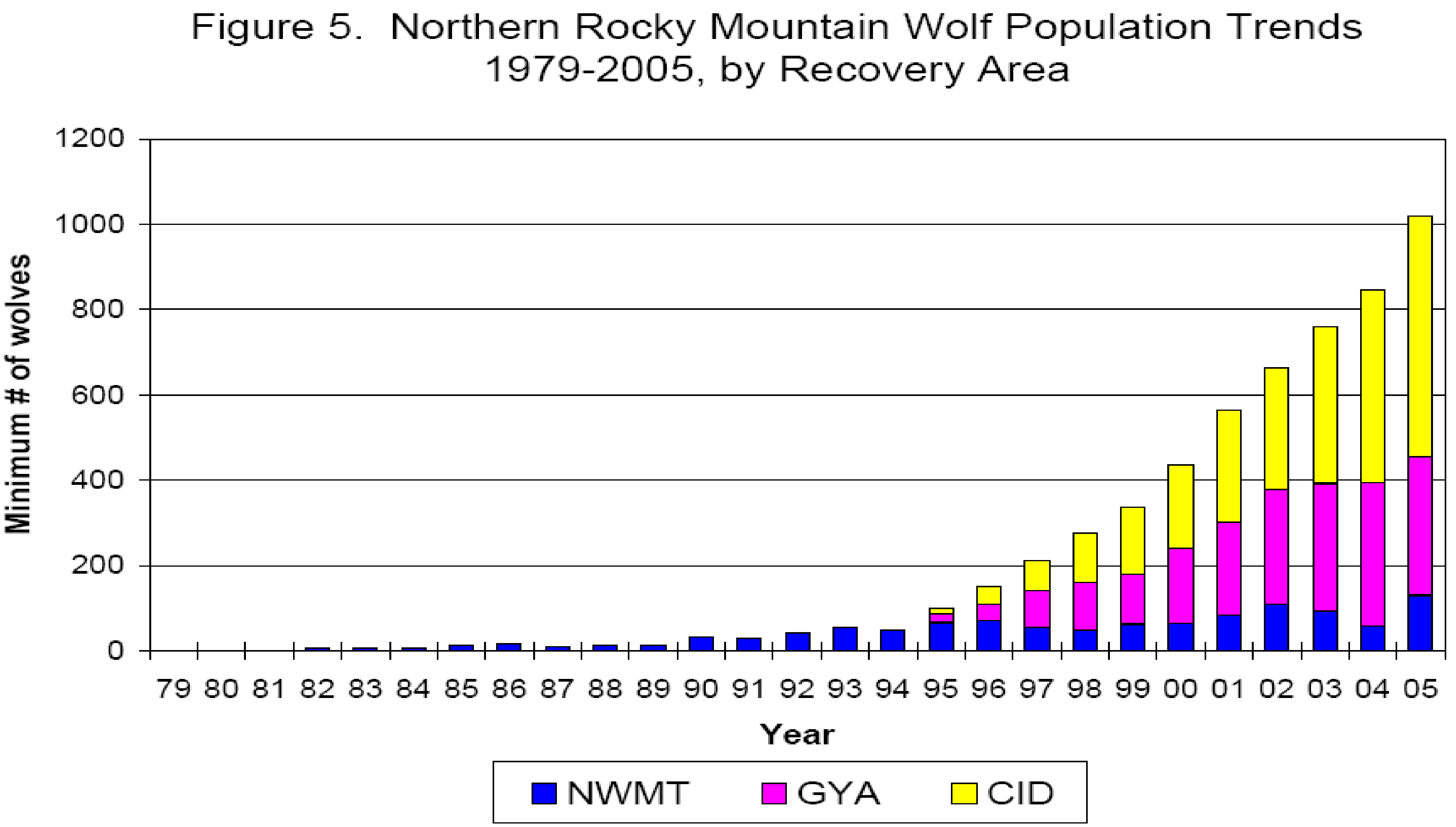
Improved Growth Model for Population Growth

Solution for r > 0 and various initial conditions.



Wolf Population in the USA

Montana Yellowstone Area Idaho



Another Example

1. Assume the growth rate is r = 1 and the carrying capacity is take K = 100. The initial condition is x(0) = 50.

The IVP is

$$x'(t) = x(1 – (x/100)) \quad x(0) = 50$$

Improved Growth Model for Population Growth

Solution to this IVP:

Divide both sides by $x(1-(x/100))$:

$$\frac{1}{x(1-(x/100))}\frac{dx}{dt}=1$$

Integrate both sides with respect to time t.

$$\int \frac{1}{x(1-(x/100))}\frac{dx}{dt}dt=\int dt$$

As before, the left hand integral can be simplified by changing the variable of integration to x:

$$\int \frac{1}{x(1-(x/100))}dx=\int dt$$

Lastly, on the left multiply numerator and denominator by 100 to get

$$\int \frac{100}{x(100-x)}dx=\int dt$$

Complete the integration of both sides.

Improved Growth Model for Population Growth

Plot of x(t)