

# Integration by Parts Applications in Engineering

by  
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# Integration by Parts - Applications in Engineering

$$\int u dv = uv - \int v du$$

- **As a technique or methodology for integrating function products, integration by parts plays a major role in all fields of engineering as its applications are commonly found in problems ranging from electric circuits, heat transfer, vibrations, structures, fluid mechanics, transport modeling, air pollution, electromagnetics, digital signal processing, and many more. To illustrate this, some representative engineering applications are detailed next.**

- **Heat Transfer Problem (EGN3358 and EML4142):** The time-dependent temperature of an object changes at a rate proportional to the difference between the temperature of its surroundings and the temperature of the object. This relation is expressed as the Newton's Law of cooling and is written as:

$$\rho c \frac{dT(t)}{dt} = - \frac{hA_s}{V} [T(t) - T_s(t)]$$

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- Where  $\rho$  is the density of the object,  $c$  is its specific heat,  $h$  is the heat transfer coefficient between the object and its surroundings,  $A_s$  is the surface area of the object,  $V$  is the volume of the object, and  $T_s(t)$  is the temperature of the surroundings. For simplicity, the equation can be expressed as:

$$\frac{dT(t)}{dt} + mT(t) = mT_s(t) \quad m = \frac{hA_s}{\rho cV}$$

- The solution of this equation can be found through the use of an integrating factor as:

$$T(t) = me^{-mt} \int T_s(t)e^{mt} dt$$

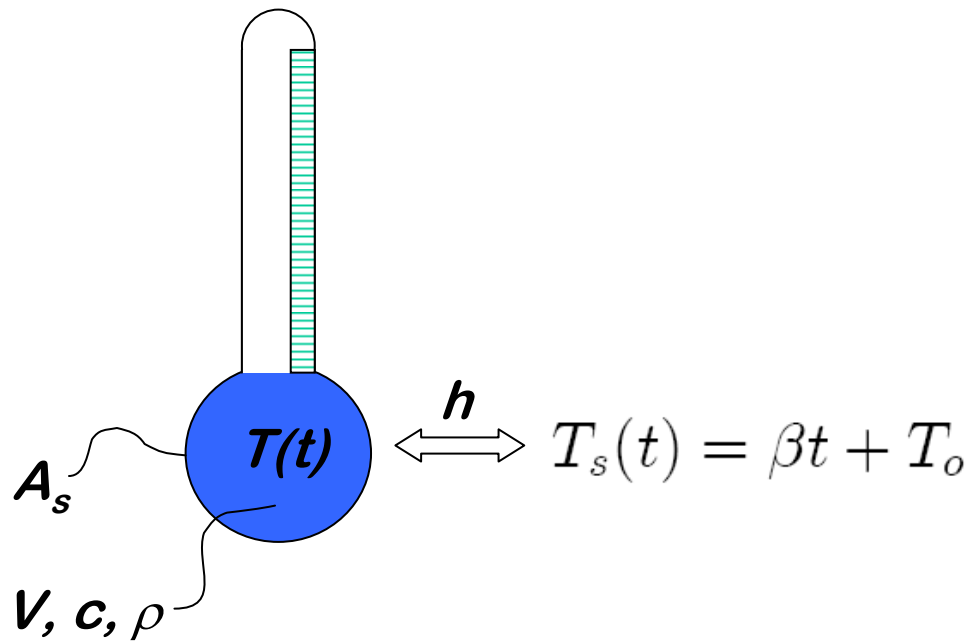
- Assume that the object is a thermometer that is being used in an experiment to read the temperature of a surrounding medium that is linearly changing in time as:

$$T_s(t) = \beta t + T_o$$

- Where  $T_o$  is the initial temperature of the surroundings and  $\beta$  is the rate at which the temperature of the surroundings is changing.

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● That is:





- Therefore, the expression for the thermometer temperature is given by:

$$T(t) = me^{-mt} \int (\beta t + T_o) e^{mt} dt$$

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● Therefore:

$$u(t) = (\beta t + T_o) \Rightarrow u'(t) = \beta$$

● And:

$$v'(t) = e^{mt} \Rightarrow v(t) = \int e^{mt} dt \Rightarrow v(t) = \frac{1}{m} e^{mt}$$

- Substitution of these terms in the integral expression leads to:

$$\int (\beta t + T_o) e^{mt} dt = \frac{1}{m} (\beta t + T_o) e^{mt} - \frac{\beta}{m} \int e^{mt} dt$$

- Solving the integral on the right-hand side and simplifying yields:

$$\int (\beta t + T_o) e^{mt} dt = \frac{1}{m} \left( \beta t + T_o - \frac{\beta}{m} \right) e^{mt} + C$$

- Substitution of this result into the expression for the temperature leads to:

$$T(t) = \left( \beta t + T_o - \frac{\beta}{m} \right) + C m e^{-mt}$$

- The constant of integration can be fixed by imposing an initial condition:

$$T(0) = T_i$$

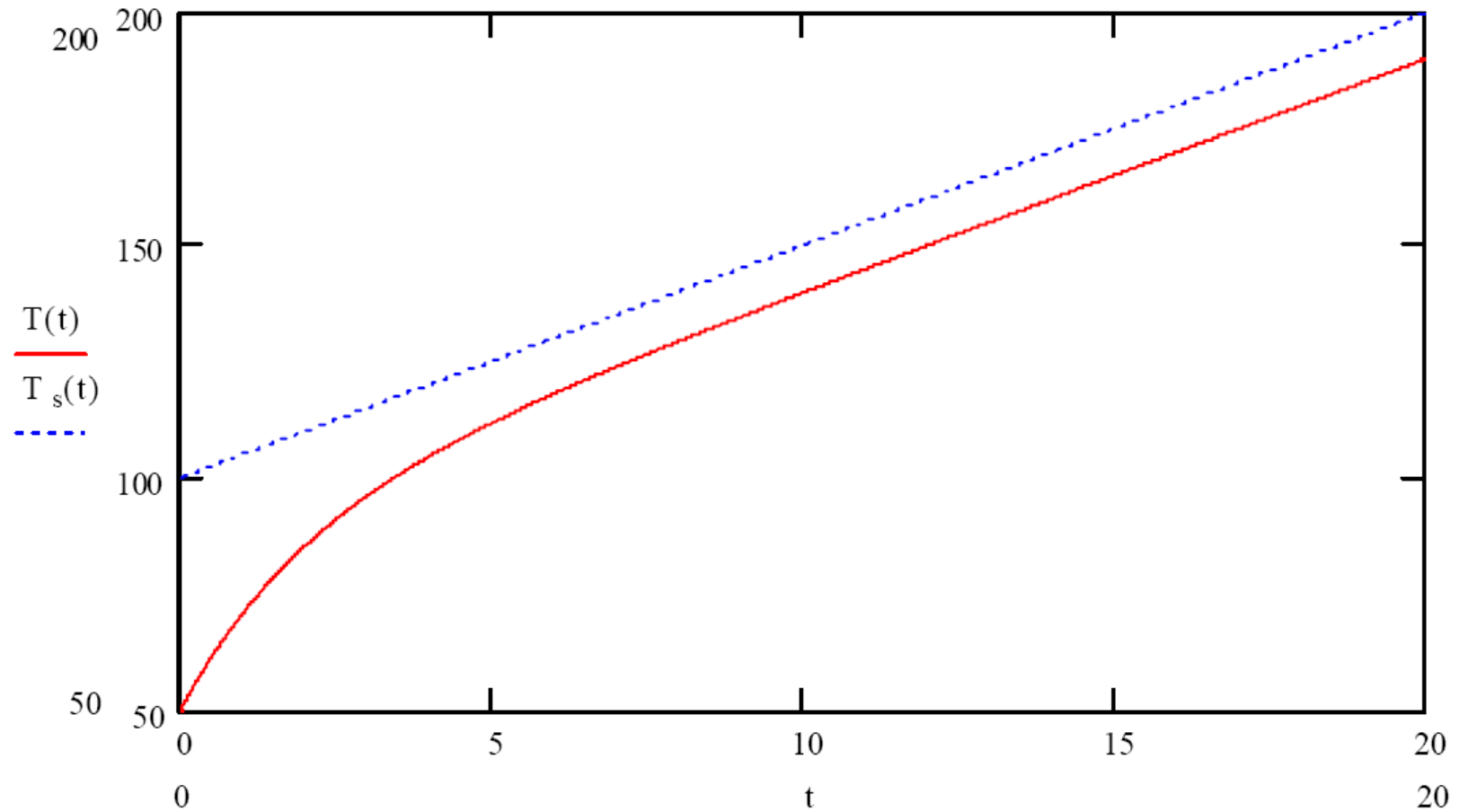
- Leading to:

$$T(t) = \left( \beta t + T_o - \frac{\beta}{m} \right) + \left( T_i - T_o + \frac{\beta}{m} \right) e^{-mt}$$

- Assume, for example, that a thermometer with a spherical test section of radius  $r=0.3\text{cm}$  ( $V = \frac{4}{3}\pi r^3$ ,  $A_s = 4\pi r^2$ ) is initially at a temperature  $T_i=50^\circ\text{C}$ , the density of the thermometer medium  $\rho=1000\text{kg/m}^3$ , its specific heat is  $c=3000\text{J/kg}^\circ\text{C}$ , the heat transfer coefficient  $h=1500\text{W/m}^2^\circ\text{C}$ , the initial temperature of the surroundings  $T_o=100^\circ\text{C}$ , and the rate of change of the surroundings temperature  $\beta=5^\circ\text{C/s}$ .

- The time-dependent temperature of the thermometer given by the solution previously described along with the surrounding temperature are provided in the plot on the next slide for the first *20* seconds.

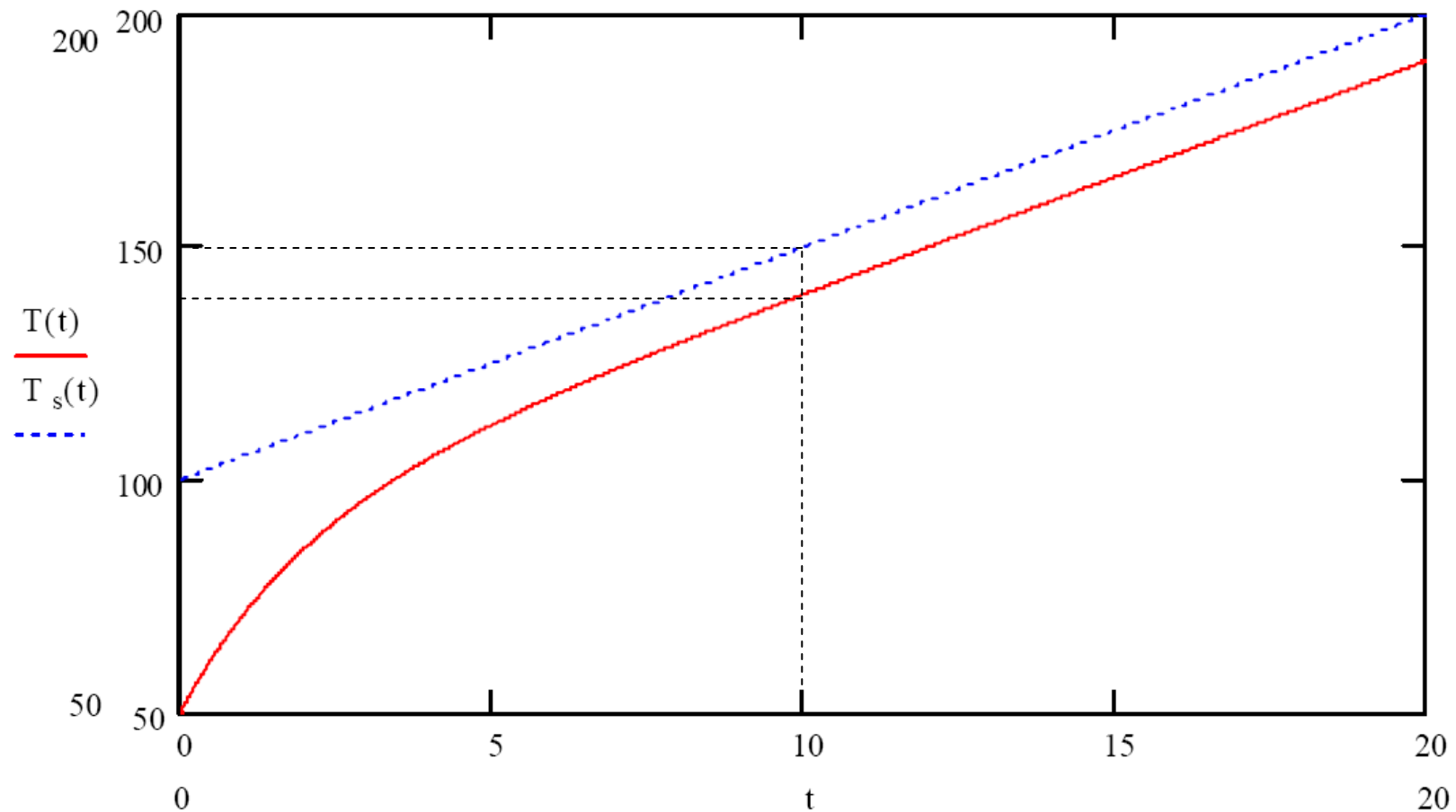
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- It can be clearly seen that the thermometer reading follows and lags behind the linearly increasing surrounding temperature at any instant in time. For instance, at  $t=10s$ , the surrounding temperature is about  $150^{\circ}\text{C}$  while the thermometer is just reading about  $140^{\circ}\text{C}$ .

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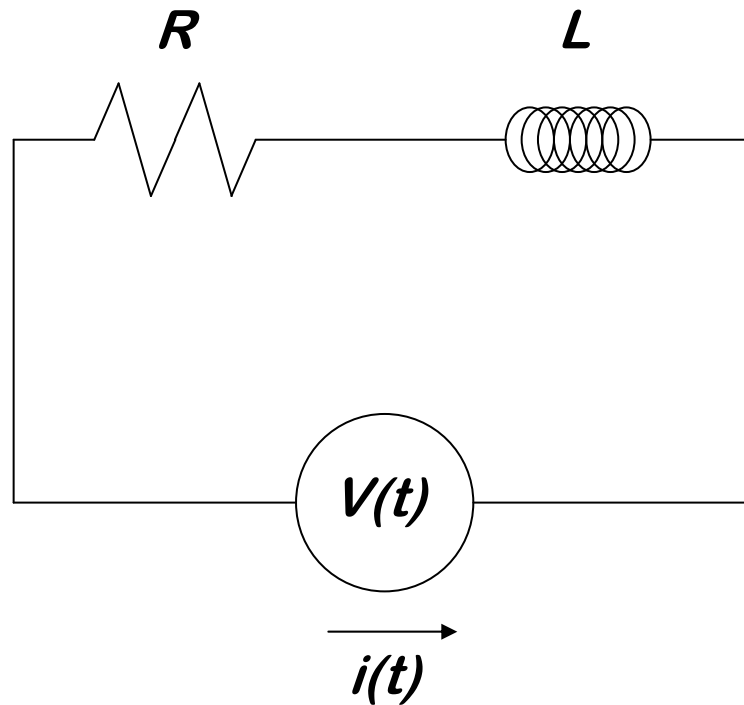


- This bias in the reading is due to the transient behavior of the surrounding temperature and can only be predicted by the analytical solution.

- **Electric Circuit Problem (EGN3373 and EEL3004):** The time-dependent current that passes through a circuit with applied voltage  $V(t)$ , constant inductance  $L$ , and constant resistance  $R$ , is described by the following equation:

$$L \frac{di(t)}{dt} + R i(t) = V(t)$$

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- Voltage  $V$ : Volt (Difference of electrical potential)
- Current  $i$ :  $1 \text{ Amp} = 1 \text{ C} / 1 \text{ s}$  (Flow of electric charge)
- Inductance  $L$ :  $1 \text{ H} = 1 \text{ Wb} / 1 \text{ Amp}$  (Magnetic flux over current)
- Resistance  $R$ :  $1 \Omega = 1 \text{ V} / 1 \text{ Amp}$  (degree of opposition to electric current, electric friction).

- The solution to the equation above can be found through the use of an integrating factor as:

$$i(t) = \frac{e^{-(R/L)t}}{L} \int V(t) e^{(R/L)t} dt$$

- Assume that an alternating voltage is applied to the circuit with a form  $V(t) = V_m \sin(\omega t)$  where  $V_m$  is the peak voltage and  $\omega$  is the alternating frequency. Then, the integral for the current takes the form:

$$i(t) = \frac{V_m}{L} e^{-(R/L)t} \int \sin(\omega t) e^{(R/L)t} dt$$



● Therefore:

$$u(t) = \sin(\omega t) \Rightarrow u'(t) = \omega \cos(\omega t)$$

● And:

$$v'(t) = e^{(R/L)t} \Rightarrow v(t) = \int e^{(R/L)t} dt \Rightarrow v(t) = \frac{L}{R} e^{(R/L)t}$$

- **Substitution leads to:**

$$\int \sin(\omega t) e^{(R/L)t} dt = \frac{L}{R} \sin(\omega t) e^{(R/L)t} - \omega \frac{L}{R} \int \cos(\omega t) e^{(R/L)t} dt$$

- **Note that the new integral on the right-hand side is not ‘simpler’ than the original one therefore it seems that integration by parts did not achieve its goal.**

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- A second integration by parts on the integral on the right-hand side will reveal an interesting feature, let:

$$u(t) = \cos(\omega t) \Rightarrow u'(t) = -\omega \sin(\omega t)$$

- And:

$$v'(t) = e^{(R/L)t} \Rightarrow v(t) = \int e^{(R/L)t} dt \Rightarrow v(t) = \frac{L}{R} e^{(R/L)t}$$

● Substitution of these new terms yields:

$$\int \sin(\omega t) e^{(R/L)t} dt = \frac{L}{R} \sin(\omega t) e^{(R/L)t} - \omega \frac{L}{R} \left[ \frac{L}{R} \cos(\omega t) e^{(R/L)t} + \omega \frac{L}{R} \int \sin(\omega t) e^{(R/L)t} dt \right]$$

## ● Collecting terms:

$$\int \sin(\omega t) e^{(R/L)t} dt = \frac{L}{R} \left[ \sin(\omega t) - \omega \frac{L}{R} \cos(\omega t) \right] e^{(R/L)t} - \omega^2 \left( \frac{L}{R} \right)^2 \int \sin(\omega t) e^{(R/L)t} dt$$

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- Note that the integral on the right-hand side is identical to the original one on the left-hand side, therefore, it can be moved to the left-hand side to yield a solution as:

$$\int \sin(\omega t) e^{(R/L)t} dt = \frac{\frac{L}{R}}{1 + \omega^2 \left(\frac{L}{R}\right)^2} \left[ \sin(\omega t) - \omega \frac{L}{R} \cos(\omega t) \right] e^{(R/L)t} + C$$

- This recursive behavior of the integration by parts implementation is typical of cases where the product of functions is composed of a trigonometric and an exponential function. Substitution of this solution into the expression for the current  $i(t)$  leads to:

$$i(t) = \frac{\frac{V_m}{R}}{1 + \omega^2 \left(\frac{L}{R}\right)^2} \left[ \sin(\omega t) - \omega \frac{L}{R} \cos(\omega t) \right] + C \frac{V_m}{L} e^{-(R/L)t}$$

- The constant of integration  $C$  can be fixed by imposing an initial condition:

$$i(0) = i_o$$

- Leading to:

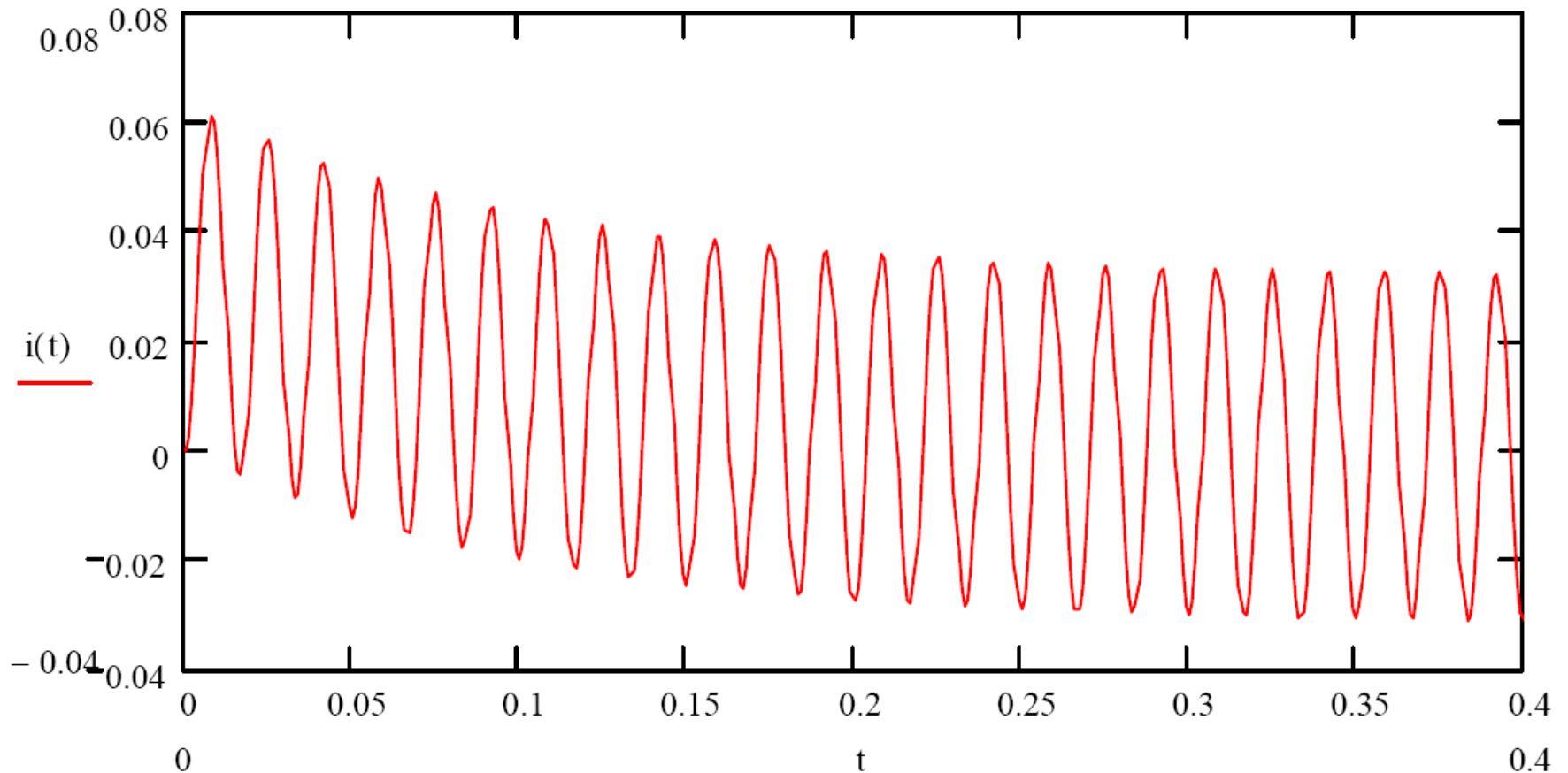
$$i(t) = \frac{\frac{V_m}{R}}{1 + \omega^2 \left(\frac{L}{R}\right)^2} \left[ \sin(\omega t) - \omega \frac{L}{R} \cos(\omega t) \right] + \left[ i_o + \frac{\omega \frac{LV_m}{R^2}}{1 + \omega^2 \left(\frac{L}{R}\right)^2} \right] e^{-(R/L)t}$$



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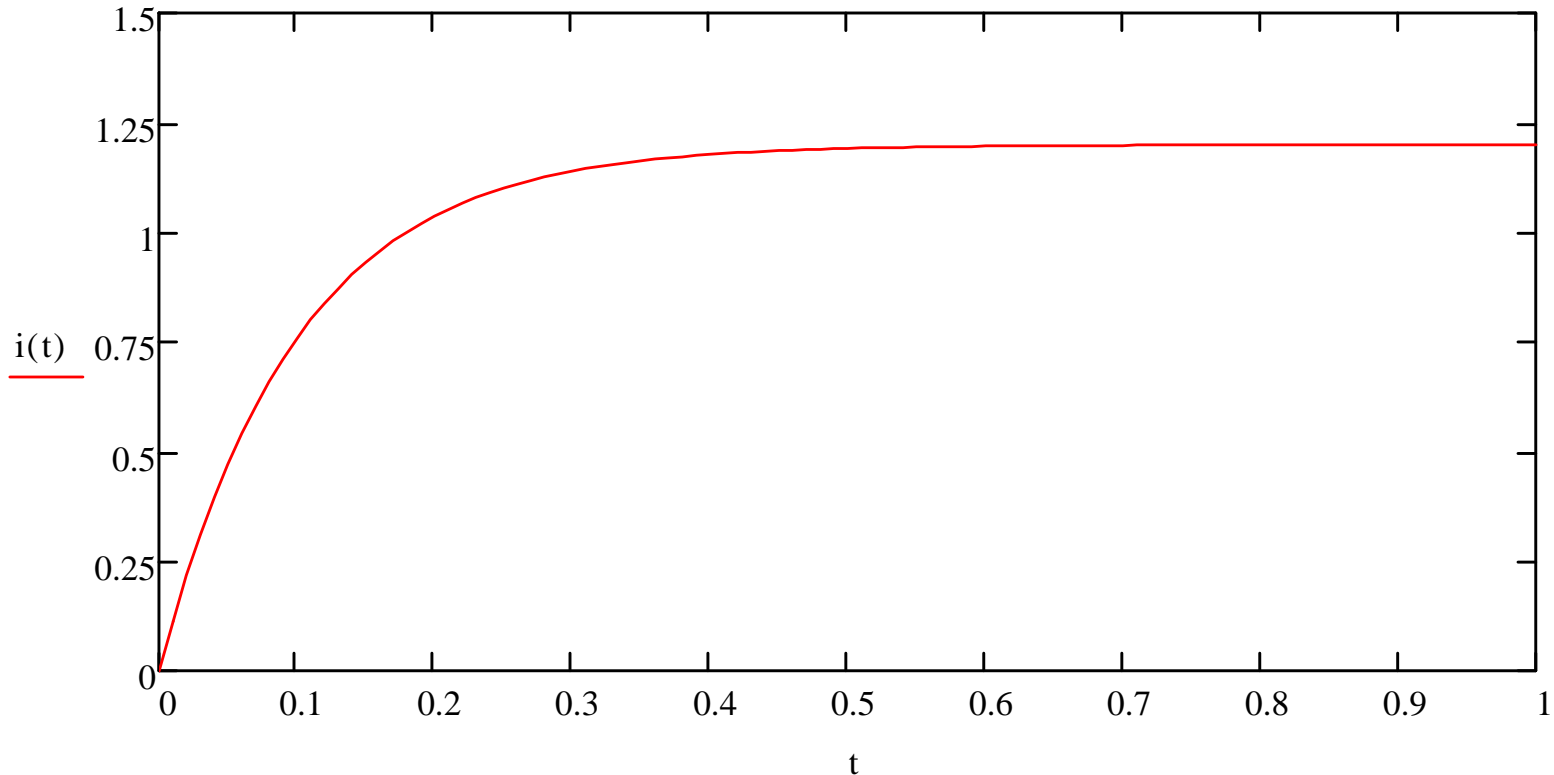
- Assume for example, that a given circuit has an applied voltage with peak value  $V_m=120$  Volts, frequency  $f=60$  Hz ( $\omega=2\pi\cdot 60$  rad/s), inductance  $L=10$  Henry, resistance  $R=100$  Ohms, and an initial current  $i_o=0$  Amps. The time-dependent current is illustrated in the plot on the next slide for the first 0.4 seconds.

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- It can be clearly seen how the oscillative current settles to a periodic alternating state after just a few fractions of a second.

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●  $i = V_m / R = 120V / 100\Omega = 1.2A$

- **Special Topics:** Integration by parts can be extended to multi-dimensional functions. That is, given the multi-dimensional function:

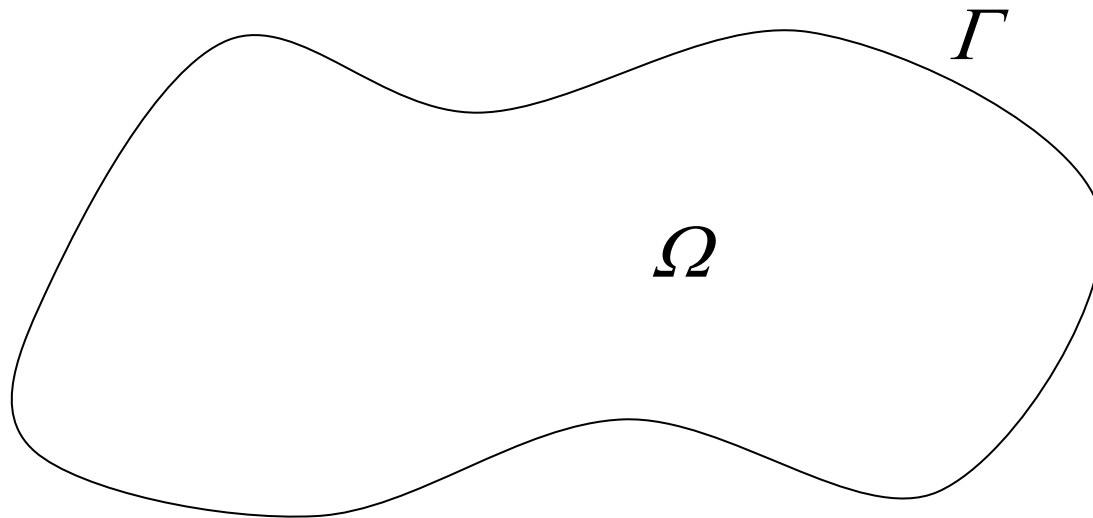
$$u(r) = u(x, y, z)$$

- **And the multi-dimensional vector field:**

$$\vec{v}(r) = \vec{v}(x, y, z)$$

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- Where  $u(r)$  and  $\vec{v}(r)$  are valid in a multi-dimensional region  $\Omega$  bounded by  $\Gamma$ :



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● Then, equivalent integration by parts is:

$$\int_{\Omega} u(r) [\nabla \cdot \vec{v}(r)] d\Omega = \oint_{\Gamma} u(r) [\vec{v}(r) \cdot \vec{n}] d\Gamma - \int_{\Omega} \vec{v}(r) \cdot \nabla u(r) d\Omega$$

● Where  $\nabla$  is the “Del” operator.

●  $\nabla \cdot \mathbf{v}$ : Divergence of vector field  $\mathbf{v}$

●  $\nabla u$ : Gradient of function  $u$

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● If we let:

$$\vec{v}(r) = \nabla f(r)$$

● Then:

$$\nabla \cdot \vec{v}(r) = \nabla \cdot \nabla f(r) = \nabla^2 f(r)$$

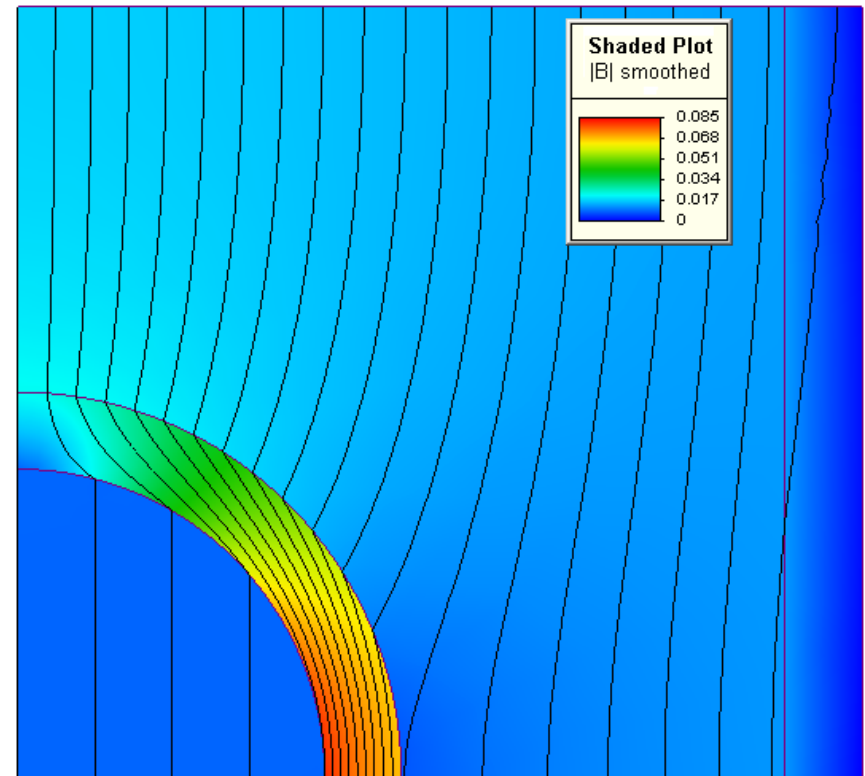
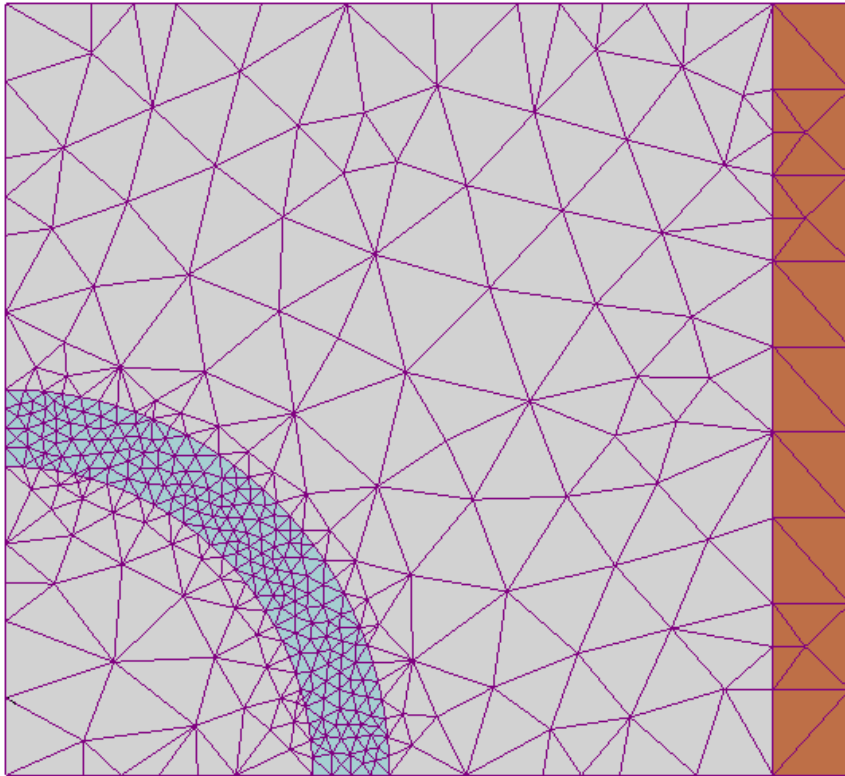


- Substitution leads to Green's first identity:

$$\int_{\Omega} u(r) \nabla^2 f(r) d\Omega = \oint_{\Gamma} u(r) [\nabla f(r) \cdot \vec{n}] d\Gamma - \int_{\Omega} \nabla f(r) \cdot \nabla u(r) d\Omega$$

- **George Green** ([14 July 1793](#) – [31 May 1841](#))

- **Green's first identity forms the basis of Finite Elements Method (FEM) analysis:**



- If the second integral on the right-hand side is integrated by parts again, then we arrive at Green's second identity:

$$\int_{\Omega} u(r) \nabla^2 f(r) d\Omega = \oint_{\Gamma} u(r) [\nabla f(r) \cdot \vec{n}] d\Gamma - \oint_{\Gamma} f(r) [\nabla u(r) \cdot \vec{n}] d\Gamma + \int_{\Omega} f(r) \nabla^2 u(r) d\Omega$$

- **Green's second identity forms the basis of the Boundary Elements Method (BEM) analysis:**

