Integration by Parts
Applications in Engineering

by

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UCF EXCEL Applications of Calculus

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$$\int u\,dv = uv - \int v\,du$$

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- **Integration by parts is a technique employed to solve integration problems of the product of two independent functions.**

$$\int x\, sin x \, dx = ?$$

$$\int t^2 e^t dt = ?$$

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- There are instances when the integral of the product of two functions can be solved directly by employing other techniques such as the general power formula:

$$\int u'(x)[u(x)]^n dx = \frac{[u(x)]^{n+1}}{n+1} + C$$

- Where the functions u'(x) and $[u(x)]^n$ appear as a product but clearly the two functions are not independent of each other as one is the derivative of the base of the other.

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- Another instance in which the product of two functions appears in an integral that can be solved directly is in the general logarithmic formula:

$$\int \frac{u'(x)}{u(x)} dx = ln|u(x)| + C$$

- Where again the functions u'(x) and 1/u(x) appear as a product but the two functions are not independent of each other.

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- The general formula for integration by parts can be easily derived from the formula for differentiation of a product of two functions as:

$$\frac{d}{dx}[u(x)v(x)] = u(x)v'(x) + v(x)u'(x)$$

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- **By integrating the left and right-hand sides of the formula above with respect to x, we find:**

$$\int \frac{d}{dx}[u(x)v(x)]dx = \int u(x)v'(x)dx + \int v(x)u'(x)dx$$

- **Recall that the integral operation is an anti-derivative therefore, the integral on the left-hand side cancels out with the derivative to yield:**

$$u(x)v(x) = \int u(x)v'(x)dx + \int v(x)u'(x)dx$$

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- **Rearranging the expression above we find:**

$$\int u(x)v'(x)dx = u(x)v(x) - \int v(x)u'(x)dx$$

- **This expression can be written in its most common and easier to remember form as:**

$$\int udv = uv - \int vdu$$

- Some mnemonics can by used to remember this formula:

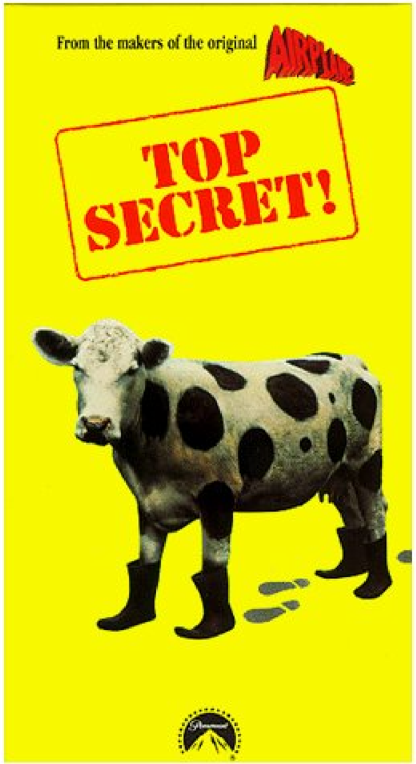
$$\int u\,dv = uv - \int v\,du$$

- Which spells out in Spanish as: *"Un Dia Vi Una Vaca Vestida De Uniforme"*

- Which translates in English as: *"One Day I Saw a Cow Dressed in Uniform"* (not so useful)

- So, in simple terms, integration by parts equates to:

$$\int u\,dv = uv - \int v\,du$$



- This expression is the general formula of integration by parts. Notice that the integral on the left-hand side contains a product of two independent functions u(x) and v'(x) and the integration by parts formula simply 'shifts' the operation so that the integral on the right-hand side is performed over the product of v(x) and u'(x) with the 'hope' that the resulting integral on the right-hand side is 'simpler' or directly solvable by any of the general formulae.

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- When confronted with an integral that seems suited for integration by parts, it is crucial to decide which of the two functions that appear as a product should be selected as u(x) and which should be selected as v'(x). For this purpose there are several rules and criteria that can be employed to achieve the goal of yielding a 'simpler' integral on the right-hand side. A general guide for function selecting when integrating by parts is known as the **ILATE** rule which can be used to decide which of the two functions in the product is the function u(x).

- **ILATE - identify the function that comes first on the following list and select it as u(x):**
  - I: inverse trigonometric functions
  - L: logarithmic functions
  - A: algebraic functions (polynomials)
  - T: trigonometric functions
  - E: exponential functions

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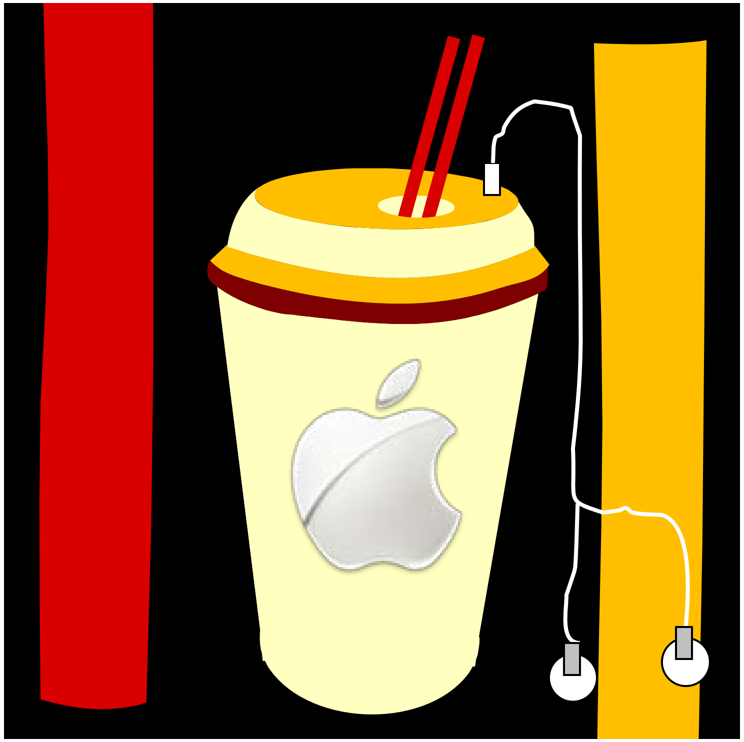
- Remember ILATE, or better yet, iLATE:



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- It should be noted that integration by parts does not guarantee a solution to the integral and the use of the **ILATE** rule just constitutes a general guide to achieve the goal of yielding a 'simpler' integral on the right-hand side.

- As a first example let us consider the following integral (Section 8.1. Example 1):

$$\int x \, sinx \, dx = ?$$

- Note that none of the general formulae (power, logarithm, etc.) can be directly implemented to solve this integral that clearly shows the power of two independent functions (x and sin x), therefore, integration by parts seems like the viable option.

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- **Therefore:**

$$u(x) = x \Rightarrow u'(x) = 1$$

- **The function v'(x) is then selected as the other part of the product as:**

$$v'(x) = sinx \Rightarrow v(x) = \int sinx\, dx \Rightarrow v(x) = -cosx$$

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- **With the explicit forms for u'(x) and v(x) solved for, the integration by parts formula can be implemented directly as:**

$$\int u\,dv = uv - \int v\,du$$

$$\int x\,\sin x\,dx = (x)\,(-\cos x) - \int (-\cos x)(1)\,dx$$

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- Rearranging:

$$\int x\, \sin x\, dx = -x \cos x + \int \cos x\, dx$$

- Notice that the goal of yielding a 'simpler' integral on the right-hand side was achieved by the proper implementation of the **ILATE** rule. Had the opposite choice been made in the function selection, the integral on the right-hand side would have ended up with a higher level of complexity than the original one on the left-hand side.

- **The integral on the right-hand side can now be solved directly leading to:**

$$\int x\, sinx\, dx = -\, x\, cosx + sinx + C$$

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- **There are cases for which it is necessary to apply the integration by parts formula more than once to yield an integral on the right-hand side that can be solved. Let us consider one of such examples (Section 8.1. Example 3):**

$$\int t^2 e^t dt = ?$$

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- **Therefore:**

$$u(t) = t^2 \Rightarrow u'(t) = 2t$$

- **And:**

$$v'(t) = e^t \Rightarrow v(t) = \int e^t dt \Rightarrow v(t) = e^t$$

- Then, substitution of these expressions into the general formula of integration by parts yields:

$$\int t^2 e^t dt = t^2 e^t - 2\int t e^t dt$$

- Notice that the integral on the right-hand side is 'simpler' than the original one, however, it is not yet in a form that can be solved directly.

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- A second integration by parts is necessary to simplify it even more. In this case:

$$u(t) = t \Rightarrow u'(t) = 1$$

- And:

$$v'(t) = e^{t} \Rightarrow v(t) = \int e^{t} dt \Rightarrow v(t) = e^{t}$$

- Notice that the **ILATE** rule was employed once again to select the functions. If the opposite selection of functions on the second integration by parts had been made, the right-hand side would have been restored as the original integral on the left-hand side.

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- **Substitution of these new expressions on the right-hand side leads to:**

$$\int t^2 e^t dt = t^2 e^t - 2\left(te^t - \int e^t dt\right)$$

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- **Solving:**

$$\int t^2 e^t dt = t^2 e^t - 2\left(te^t - e^t + C\right)$$

- **Rearranging:**

$$\int t^2 e^t dt = \left(t^2 - 2t + 2\right)e^t + C_1$$

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- **The integration by parts formula can also be implemented for definite integrals by simply transferring the limits of integration to the right-hand side as:**

$$\int_a^b u(x)v'(x)dx = u(x)v(x)\Big|_a^b - \int_a^b v(x)u'(x)dx$$

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- **Let us consider the following example:**

$$\int_1^e ln x \, dx = ?$$

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- **Therefore:**

$$u(x) = lnx \Rightarrow u'(x) = \frac{1}{x}$$

- **And:**

$$v'(x) = 1 \Rightarrow v(x) = \int 1dx \Rightarrow v(x) = x$$

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- Then, substitution of these expressions into the general formula of integration by parts yields:

$$\int_1^e \ln x \, dx = x \ln x \Big|_1^e - \int_1^e x \left(\frac{1}{x}\right) dx$$

- Solving and applying the limits of integration:

$$\int_1^e \ln x \, dx = (x \ln x - x)\Big|_1^e = (e \ln e - e) - (1 \ln 1 - 1) = 1$$

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