Applications of Calculus I

Application of Maximum and Minimum Values and Optimization to Engineering Problems

by
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Outline

• Review of Maximum and Minimum Values in Calculus
• Review of Optimization
• Applications to Engineering
Maximum and Minimum Values

- You have seen these in Chapter 4
- Some important applications of differential calculus need the determination of these values
- Typically this involves finding the maximum and/or minimum values of a Function
- Two Types – Global (or Absolute) or Local (or Relative).
Local Maxima or Minima

- Fermat’s Theorem – If a function \( f(x) \) has a local maximum or minimum at \( c \), and if \( f'(c) \) exists, then
  \[
  f'(c) = 0
  \]

- Critical Number \( c \) of a function \( f(x) \) is a number such that either
  \[
  f'(c) = 0
  \]
or it does not exist.
Closed Interval Method

• Used to find the Absolute (Global) Maxima or Minima in a Closed Interval \([a,b]\)
  – Find \(f\) at the critical numbers of \(f\) in \((a,b)\)
  – Find \(f\) at the endpoints
  – Largest value is absolute maximum and smallest is the absolute minimum
Engineering - Demo

• [http://www.funderstanding.com/k12/coaster/](http://www.funderstanding.com/k12/coaster/)

• Highlights the importance of the following:
  – Understanding of Math
  – Understanding of Physics
  – Influence of Several Independent Variables
  – Fun
Calculus Application – Graphing and Finding Maxima or Minima

Section 4.1 #66:

On May 7, 1992, the space shuttle Endeavor was launched on mission STS-49, the purpose of which was to install a new perigee kick motor in an Intelsat communications satellite. The table gives the velocity data for the shuttle between liftoff and the jettisoning of the solid rocket boosters.
Calculus Application – Graphing and Finding Maxima or Minima

<table>
<thead>
<tr>
<th>Event</th>
<th>Time (s)</th>
<th>Velocity (ft/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Launch</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Begin roll maneuver</td>
<td>10</td>
<td>185</td>
</tr>
<tr>
<td>End roll maneuver</td>
<td>15</td>
<td>319</td>
</tr>
<tr>
<td>Throttle to 89%</td>
<td>20</td>
<td>447</td>
</tr>
<tr>
<td>Throttle to 67%</td>
<td>32</td>
<td>742</td>
</tr>
<tr>
<td>Throttle to 104%</td>
<td>59</td>
<td>1325</td>
</tr>
<tr>
<td>Maximum dynamic pressure</td>
<td>62</td>
<td>1445</td>
</tr>
<tr>
<td>Solid rocket booster separation</td>
<td>125</td>
<td>4151</td>
</tr>
</tbody>
</table>
Shuttle Video
Calculus Application – Graphing and Finding Maxima or Minima

• Use a graphing calculator or computer to find the cubic polynomial that best models the velocity of the shuttle for the time interval \(0 \leq t \leq 125\). Then graph this polynomial.

• Find a model for the acceleration of the shuttle and use it to estimate the maximum and minimum values of acceleration during the first 125 seconds.
Strategy!

- Let us use a computer program (MS-EXCEL) to graph the variation of velocity with time for the first 125 seconds of flight after liftoff.
- The graph is first created as a scatter plot and then a trendline is added.
- The trendline menu allows for the selection of a polynomial fit and a cubic polynomial is picked as required in the problem description above.
Shuttle Velocity Profile

\[ y = 0.0015x^3 - 0.1155x^2 + 24.982x - 21.269 \]
Solution

• From the graph, the function $y(x)$ or $v(t)$ can be expressed as

\[ v(t) = 0.0015t^3 - 0.1155t^2 + 24.982t - 21.269 \]

• Acceleration is the derivative of velocity with time.

\[ a(t) = \frac{dv(t)}{dt} = 0.0045t^2 - 0.231t + 24.982 \]
Solution Continued

• During the first 125 seconds of flight, that is in the interval $0 \leq t \leq 125$; apply the Closed Interval Method to the continuous function $a(t)$ on this interval. The derivative is

$$a'(t) = \frac{da(t)}{dt} = 0.009t - 0.231$$

• The critical number occurs when \( a'(t) = 0; \)

which gives us

$$t_1 = \frac{0.231}{0.009} \approx 25.67$$

seconds.
Solution Continued

• Evaluating the acceleration at the Critical Number and at the Endpoints, we get

\[ a(25.67) = 22.0 \text{ ft/s}^2 \]

\[ a(0) = 24.982 \text{ ft/s}^2 \]

\[ a(125) = 66.42 \text{ ft/s}^2 \]

• Thus, the maximum acceleration is 66.42 ft/s\(^2\) and the minimum is 22.0 ft/s\(^2\).
Calculus Application – Optimization

Section 4.7 #34:

• A fence is 8 feet tall and runs parallel to a tall building at a distance of 4 feet from the building.

• What is the length of the shortest ladder that will reach from the ground over the fence to the wall of the building?
Calculus Application – Optimization

![Diagram of a ladder leaning against a building with a fence in between, showing the optimization problem involving angles and distances.]
Calculus Application – Strategy

• From the figure using trigonometry, the length of the ladder can be expressed as

\[ L = AB + BC = \frac{H}{\sin \theta} + \frac{D}{\cos \theta} \]

• Next, find the critical number for \( \theta \) for which the length \( L \) of the ladder is minimum.

• Differentiating \( L \) with respect to \( \theta \) and setting it equal to zero.
Engineering Courses with Math

• Some future Engineering Courses at UCF that you may take are
  – **EGN3310** – Engineering Mechanics – Statics
  – **EGN3321** – Engineering Mechanics – Dynamics
  – **EGN 3331** – Mechanics of Materials
  – **EML 3601** – Solid Mechanics

• and several of your engineering major courses
Use of Calculus in Engineering

• Real-world Engineering Applications that use Calculus Concepts such as Derivatives and Integrals

• Global and Local Extreme Values are often needed in optimization problems such as
  – Structural or Component Shape
  – Optimal Transportation Systems
  – Industrial Applications
  – Optimal Biomedical Applications
Calculus Topics Covered

- Global and local extreme values
- Critical Number
- Closed Interval Method
- Optimization Problems using Application to Engineering Problems
Applications to Engineering

• Maximum Range of a projectile – (Mechanical and Aerospace engineering)
• Optimization of Dam location on a River (Civil engineering)
• Potential Energy and Stability of Equilibrium (Mechanical, Civil, Aerospace, Electrical Engineering)
Applications to Engineering

• Optimal Shape of an Irrigation Channel (Civil engineering)
• Overcoming Friction and other Forces to move an Object (Mechanical, Aerospace, Civil engineering)
Application to Projectile Dynamics

- Maximum Range for a Projectile
- May also be applied to Forward Pass in Football
- Goal 1: To find the Maximum Range $R$ of a projectile with Muzzle (Discharge) Velocity of $v$ meters/sec
- Goal 2: Find Initial Angle of Elevation to achieve this range
Engineering Problem Solution

• Gather All Given Information
• Establish a Strategy for the Solution
• Collect the Tools (Concepts, Equations)
• Draw any Figures/Diagrams
• Solve the Equations
• Report the Answer
• Consider – Is the answer Realistic?
Given Information

• The Range $R$ is a function of the muzzle velocity and initial angle of elevation:

$$R = \frac{v^2 \sin 2\theta}{g}$$

• $\theta$ is the angle of elevation in radians and $g$ is the acceleration due to gravity equal to 9.8 m/s$^2$
Strategy!

- We need to find the maximum value of the range $R$ with respect to different angles of elevation.
- Differentiate $R$ with respect to $\theta$ and set it to zero to find the global maxima. Note that in this case, $\nu$ and $g$ are constants.
- The end points for the interval for forward motion are $0 \leq \theta \leq \frac{\pi}{2}$.
Solution

\[ R = \frac{v^2 \sin 2\theta}{g} \]

\[ \frac{dR}{d\theta} = \frac{v^2 (2 \cos 2\theta)}{g} = 0 \]

As \( v \) and \( g \) are both non-zero,

\[ \cos 2\theta = 0 \]

Using trigonometric double angle formula:

\[ \cos 2\theta = 2 \cos^2 \theta - 1 = 0 \]

\[ \cos \theta = \frac{1}{\sqrt{2}} \]

or,

\[ \theta = \frac{\pi}{4} \]
Solution Continued

Evaluating the range at the Critical Value gives

$$R\left(\frac{\pi}{4}\right) = \frac{v^2}{g}$$

And At the End Points
$$R(0) = 0$$
$$R(\pi/2) = 0$$

Maximum range for the projectile is reached when

$$\theta = \frac{\pi}{4}$$ or 45°
Optimizing the Shape of Structures

• Relates to Fluid Mechanics and Hydraulics in Civil Engineering
• Civil Engineers have to design Hydraulic Systems at Optimal Locations along Rivers
• They also have to Optimize the Size of the Dam for Cost Constraints
Optimal Location of Dam

Depth of Water: \[ D(x) = 20x + 10 \]

Width of River: \[ W(x) = 10(x^2 - 8x + 22) \]
Example of a Dam on a River
Given Constraints and Questions

• If the dam cannot be more than 310 feet wide and 130 feet above the riverbed, and the top of the dam must be 20 feet above the present river water surface, what is a range of locations that the dam can be placed (A)?

• What are the dimensions of the widest and narrowest dam (B) that can be constructed in accordance with the above constraints?

• If the cost is proportional to the product of the width and the height of the dam, where should the most economical dam be located (C)?
Strategy!

• Use the Closed Interval Method to find the widest and narrowest dam in the range of acceptable locations of the dam.
• Define the Cost Function as proportional to the product of width and height
• Minimize Cost Function with respect to the location $x$ measured from Rock Springs
Solution (A)

Based on the Specified Constraints
Width must be less than 310:
\[ W(x) = 10(x^2 - 8x + 22) \leq 310 \Rightarrow -1 \leq x \leq 9 \]

Depth must be less than 110
\[ D(x) = 20x + 10 \leq 110 \Rightarrow x \leq 5; \]

Range of locations for the Dam
\[ 0 \leq x \leq 5 \]
Solution (B)

To obtain the widest (maximum W) and narrowest (minimum W) for the dam, apply the Closed Interval Method for the function $W(x)$ in the interval $0 \leq x \leq 5$

$$W(x) = 10(x^2 - 8x + 22)$$

Differentiating:

$$\frac{dW(x)}{dx} = 20x - 80 = 0$$

Critical Value:

$$x = 4$$
Solution (B) Continued

Corresponding width \( W(4) = 60 \) feet is the Minimum Width.
Next, checking the endpoints of the interval, we obtain the following values:

\[
W(5) = 70 \text{ feet and } W(0) = 220 \text{ feet}
\]

Maximum Width of the dam is 220 feet at Rock Spring \((x = 0)\).
Solution (C) Cost Minimization

Height of Dam must be 20 feet $\text{HIGHER}$ than Depth of Water there -

\[ H(x) = D(x) + 20 = 20x + 30 = 10(2x + 3) \]

Cost Function is Proportional to Product of $H$ and $W$

\[ C(x) = F(x^2 - 8x + 22)(2x + 3) \]

Where $F$ is a positive Constant; Simplifying -

\[ C(x) = F(2x^3 - 13x^2 + 20x + 66) \]
Solution (C) Continued

To Find the Critical Number -

\[
\frac{dC(x)}{dx} = 0 \quad \text{or} \quad \frac{dC(x)}{dx} = 2F(3x - 10)(x - 1) = 0
\]

Solving for two values of \( x \) -

\[ x = \frac{10}{3} \]

Cheaper Dam is at \( x = \frac{10}{3} \)

Cost of Dam at this location = $62.30F

Checking Endpoints – at \( x = 0 \), Cost = $66F
and at \( x = 5 \), Cost = $91F

MINIMUM COST = $62.30F at \( x = \frac{10}{3} \)
My Current Research Areas

- Permeable Concrete Pavements
- Soil Erosion and Sediment Control
- Slope Stability of Soil Structures and Landfills
- Modeling of Structures – Pile Foundations
Permeable Concrete Pavements
Optimization of Water Transport Channel

- Applies to Land Development and Surface Hydrological Engineering
- Such applications are common in Water and Geotechnical areas of Civil Engineering
- Part of the Overall Design of the Irrigation Channel – other areas Structural design, Fluid Flow Calculations and Location
Irrigation Water Transport Channel
Objective

A trapezoidal channel of uniform depth $d$ is shown below. To maintain a certain volume of flow in the channel, its cross-sectional area $A$ is fixed at say 100 square feet. Minimize the amount of concrete that must be used to construct the lining of the channel.

$\theta$ is the angle of inclination of each side. The other relevant dimensions are labeled on the figure.
Strategy!

- Make Simplifying Assumptions (at this level) –
  \[ \theta_1 = \theta_2 = \theta, \quad \text{and} \quad e_1 = e_2 = e. \]

- Minimize the Length L of the Channel Perimeter excluding the Top (surface) Length
Solution

Based on Geometry: \[ L = h_1 + b + h_2 \]

\[ h_1 = h_2 = \frac{d}{\sin \theta} \]

Since the Cross-sectional Area of the Channel must = 100 sq ft

\[ A = 100 = bd + 2 \left[ \frac{1}{2} ed \right] \]

or, \[ b = \frac{100}{d} - e = \frac{100}{d} - \frac{d}{\tan \theta} \]
Wetted length (Length in contact with water when full)

\[
L = \frac{100}{d} - \frac{d}{\tan \theta} + \frac{2d}{\sin \theta}
\]

Minimizing \( L \) as a function of \( \theta \) and \( d \) requires advanced multivariable calculus. To simplify, let us make a DESIGN ASSUMPTION. Assume one of the two variables -

\[
\theta = \frac{\pi}{3}
\]
Solution - Continued

Expression for $L$ now is -

$$L = f(d) = \frac{100}{d} - \sqrt{3}d$$

To get Global Minimum for $L$ for

$$0 < d < \infty$$

$$\frac{dL}{dd} = f'(d) = -100d^{-2} + \sqrt{3} = 0$$

or, $$d^2 = \frac{100}{\sqrt{3}}$$

or, $$d = \frac{10}{\frac{4}{\sqrt{3}}} = 7.5984.$$
Solution - Continued

Since

\[ f''(d) = \frac{200}{d^3} > 0 \]

in the interval \((0, \infty)\)

Length of the Channel with \(d = 7.5984\)

\[ f(7.5984) \approx 26.322 \]

is the Global Minimum
Soil Erosion Test Laboratory
Minimizing Energy to Build Stable Systems

- Applies to both Mechanical and Civil Engineers
- Potential Energy is encountered in Mechanics and in Machine Design and Structural Analysis
- Minimizing Potential Energy maintains Equilibrium State and helps in Stability
Example - Pinned Machine Part
Given Information

Pinned Bars form the parts of a Machine

Held in place by a Spring

Each Bar weighs $W$ and has a length of $L$

Spring is UNSTRETCHED when $\alpha = 0$ and in equilibrium when $\alpha = 60^\circ$
Objective

• Find the value of the Spring Constant such that the system is in Equilibrium

• Determine if this Equilibrium Position is Stable or Unstable?
Strategy!

- Note that the “forces” that do the work to generate potential energy are
  - Weight of the Bars
  - Force in the Spring pulling to the right

- Express Potential Energy $U$ as a function of the angle $\alpha$ and solve for $k$ using
  \[
  \frac{dU}{d\alpha} = 0
  \]
Solution

Setting the Reference State or Datum at A –

Potential Energy = Sum of (Weight of each Bar times the Translations or movement of each Bar)
Solution - continued

Due to the two Bars – potential energy is

\[ U_1 = W \left( -\frac{1}{2} L \sin \alpha \right) + W \left( -\frac{1}{2} L \sin \alpha \right) = -WL \sin \alpha \]

Change in spring length or stretch of spring

\[ \delta = 2L - 2L \cos \alpha \]

Potential Energy due to Spring

\[ U_2 = \frac{1}{2} k (2L - 2L \cos \alpha)^2 \]
Solution - continued

Total Potential Energy becomes

\[ U = U_1 + U_2 \]

or

\[ U = -WL \sin \alpha + 2kL^2 (1 - \cos \alpha)^2 \]

When in Equilibrium state, Total U is in a MINIMUM state with respect to the rotation from REST STATE

\[ \frac{dU}{d\alpha} = 0 \]
Solution - continued

Differentiating and setting equal to 0 -

\[
\frac{dU}{d\alpha} = -WL \cos \alpha + 4kL^2 \sin \alpha (1 - \cos \alpha) = 0
\]

Given that the angle is \(\alpha = 60^\circ\)

Solving for \(k\) –

\[
k = \frac{W \cos \alpha}{4L \sin \alpha (1 - \cos \alpha)} = \frac{W \cos 60}{4L \sin 60 (1 - \cos 60)} = \frac{0.289L}{W}
\]
Stability Check

Second Derivative of Potential Energy is an \textbf{INDICATOR} of Stability of the System

\[
\frac{d^2U}{d\alpha^2} = WL \sin \alpha + 4kL^2 (\cos \alpha - \cos^2 \alpha + \sin^2 \alpha)
\]

\[
= WL \sin 60 + 4kL^2 (\cos 60 - \cos^2 60 + \sin^2 60)
\]

As \( W, L \) and \( k \) are positive quantities, the Equilibrium Position at \( \alpha = 60^\circ \) is \textbf{STABLE}!
Application to Beams

- Design of Beams requires knowledge of forces inside the beam
- Two types – Shear and Bending Moment
- Design engineers PLOT the distribution along the beam axis
- Use Derivatives to determine Maximum and Minimum values and other parameters
Application of Calculus to Friction and Static Equilibrium Problem

• Friction is Important for Different Areas of Engineering – ME, AE, CE, and IE
• This example deals with a Concept you will see shortly in Engineering Mechanics Class
• Concepts Include – Free-body Diagrams, Friction, Newton’s Laws (3rd) and Equations of Equilibrium
Forces Needed to Move a Stuck Car

- Man exerts a force $P$ on the Car at an angle of $\alpha$

- Car is Front Wheel Drive with Mass = 17.27 kN

- Driver in Car is able to Spin the Front Wheels $\mu_k = 0.02$

- Snow behind the back tires has built up and exerts a Force = $S$ kN
Objective

• Getting the Car UNSTUCK and moving requires Overcoming a Resisting Force of \( S = 420 \) N

• What angle \( \alpha \) minimizes the force \( P \) needed to overcome the resistance due to the snow
Strategy!

• Draw Pictorial Representation of ALL forces on the Car – Free-Body Diagram (FBD)
• Apply Equations of Equilibrium to this FBD (will learn in PHY and use in EGN Classes)
• Express P as a function of angle of push $\alpha$
• Find the Global Minimum for P in the range $0 < \alpha < 90^\circ$
Free-body Diagram
Solution

• Equations of Equilibrium are applied to the FBD

• This implies the BALANCE of all the FORCES and MOMENTS (Rotations) on the System
Equations of Equilibrium

\[ S - \mu_k N_F - P \cos \alpha = 0 \]
\[ N_R + N_F - W - P \sin \alpha = 0 \]
\[ -W (1.62) + N_F (2.55) + P \cos \alpha (0.90) - P \sin \alpha (3.40) = 0 \]

\[ N_F = \frac{1}{\mu_k} (S - P \cos \alpha) \]

\[ N_R = -N_F + W + P \sin \alpha \]
Expression for Force P (angle)

\[-1.62W + 2.55 \frac{1}{\mu_k} (S - P \cos \alpha) + 0.90P \cos \alpha - 3.40P \sin \alpha = 0\]

Differentiating, using the Chain Rule to find \( \frac{dP}{d\alpha} \)

and setting it equal to 0 gives us the minimum value (critical) of \( \alpha \)
Computations

\[ \frac{2.55}{\mu_k} \left[ -\frac{dP}{d\alpha} \cos \alpha + P \sin \alpha \right] + 0.90 \frac{dP}{d\alpha} \cos \alpha - 0.90P \sin \alpha - 3.40 \frac{dP}{d\alpha} \sin \alpha - 3.40P \cos \alpha = 0 \]

\[ \frac{dP}{d\alpha} = \frac{-P \left[ \left( \frac{2.55}{\mu_k} - 0.90 \right) \sin \alpha - 3.40 \cos \alpha \right]}{0.90 \cos \alpha - 3.40 \sin \alpha - \left( \frac{2.55}{\mu_k} \right) \cos \alpha} = 0 \]
Minimum Value of Angle of Push

\[ \tan \alpha = \frac{3.40}{\left( \frac{2.55}{\mu_k} - 0.90 \right)} \]

or, \( \alpha = 1.54° \)
The End

• You now know more about how Differential Calculus is used in Engineering!
• Good Luck!

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