Calculus Topic: Integration of Rational Functions

Section 8.4 # 10: Evaluate the integral

$$\int \frac{1}{(t+4)(t-1)} dt$$

Solution: The denominator of the integrand is already factored with the factors being distinct, so

$$\frac{1}{(t+4)(t-1)} = \frac{A}{t+4} + \frac{B}{t-1}.$$

When the right hand side is recombined with a common denominator we have

$$\frac{1}{(t+4)(t-1)} = \frac{A}{t+4} + \frac{B}{t-1} = \frac{A(t-1)+B(t+4)}{(t+4)(t-1)}.$$

This means that

$$1 = A(t-1) + B(t+4)$$

which must be true for all values of t. So when $t = 1$, we have 1$= B(1+4) = 5B$ and then $B = \frac{1}{5}$. Similarly when $t = -4$, we have $1 = A(-4-1) = -5A$ which means $A = -\frac{1}{5}$. The integral is now

$$\int \left(\frac{-\frac{1}{5}}{t+4} + \frac{\frac{1}{5}}{t-1}\right) dt = -\frac{1}{5}\int \frac{dt}{t+4} + \frac{1}{5}\int \frac{dt}{t-1}$$

Question1: What is the name of the property that allows us to write down the right hand side?

1. associative property for integration
2. commutative property for integration
3. linearity property for integration

These integrals are then evaluated using the natural log function. The result is

$$\int \frac{1}{(t+4)(t-1)} dt = \frac{1}{5}(\ln|t-1| - \ln|t+4|) + C = \frac{1}{5}\ln\left|\frac{t-1}{t+4}\right| + C$$

where of course it is very important that we include the integration constant C.

Example 2: Evaluate the integral

$$\int \frac{2t}{3t^2 + 10t + 3} dt$$

Step 1: Factor the denominator.

1. $(3t - 1)(t - 3)$
2. $(3t + 1)(t - 3)$
3. $(3t + 1)(t + 3)$
4. $(3t - 1)(t + 3)$

Step 2: The correct factoring is 3) $(3t+1)(t+3)$. Since each factor is distinct the partial fraction expansion is of the form:

$$\frac{2t}{3t^2 + 10t + 3} = \frac{A}{3t+1} + \frac{B}{t+3}.$$

When the right hand side is recombined with a common denominator we have

$$\frac{2t}{3t^2 + 10t + 3} = \frac{A}{3t+1} + \frac{B}{t+3} = \frac{A(t+3) + B(3t+1)}{(3t+1)(t+3)}.$$

This means that

$$2t = A(t+3) + B(3t+1)$$

The value of A is

1. $A = 1/4$
2. $A = -1/4$
3. $A = 1/3$
4. $A = -1/3$

The value of B is

1. $B = 2/3$
2. $B = -2/3$
3. $B = 3/4$
4. $B = -3/4$

Step 3: The correct values are $A = -1/4$ and $B = 3/4$. The integral is now

$$\int \left(\frac{-\frac{1}{4}}{3t+1} + \frac{\frac{3}{4}}{t+3}\right) dt = -\frac{1}{4}\int \frac{dt}{3t+1} + \frac{3}{4}\int \frac{dt}{t+3}$$

The correct anti-derivative is

1. $-\frac{1}{4}\ln|3t+1| + \frac{3}{4}\ln|t+3| + C$
2. $-\frac{3}{4}\ln|3t+1| + \frac{3}{4}\ln|t+3| + C$
3. $-\frac{1}{12}\ln|3t+1| + \frac{3}{4}\ln|t+3| + C$
4. $-\frac{1}{4}\ln|3t+1| + \frac{1}{4}\ln|t+3| + C$

Background

In biology, mathematical models have been used for many years to describe the population of a particular species, whether it be human, bacteria or an endangered species.

A mathematical model for the population model should:

- Find qualitative behavior
- Find quantitative behavior

Basic assumptions:

- deterministic - the future determined by the present
- population of a given species can be represented by a continuous, differentiable function of time $x(t)$. This is a good approximation if the population is large.
- population is in isolation - no predators or competitors

One of the simplest way to model a population is with a differential equation (DE). A differential equation is an equation where

- the unknown is a function that must be found
- there are terms that include the effect of at least one derivative

An example of a DE with unknown $x(t)$ that also contains the first derivative is

$$x'(t) = 4x(t) + t^2$$

A famous example of a differential equation is Newton's second law:

$$mx''(t) = F(x(t), x'(t), t)$$

where m is the mass and F is the applied force.

What is a Solution?

A solution to a differential equation is one that on substitution satisfies the equation. In general the solution to a differential equation is an infinite family of functions. To choose one of these functions to be the solution, require that the solution pass through a particular point. That is

$$x(t_0) = x_0$$

where t_0 and x_0 are constants. An example might be

$$x(4) = 20$$

This is called an initial condition.

Solution methods

- analytical techniques - integration, transform methods, symmetry methods
- numerical methods - finite difference

The subject of differential equations is a rich one but we will focus on examples that can be solved by integration.

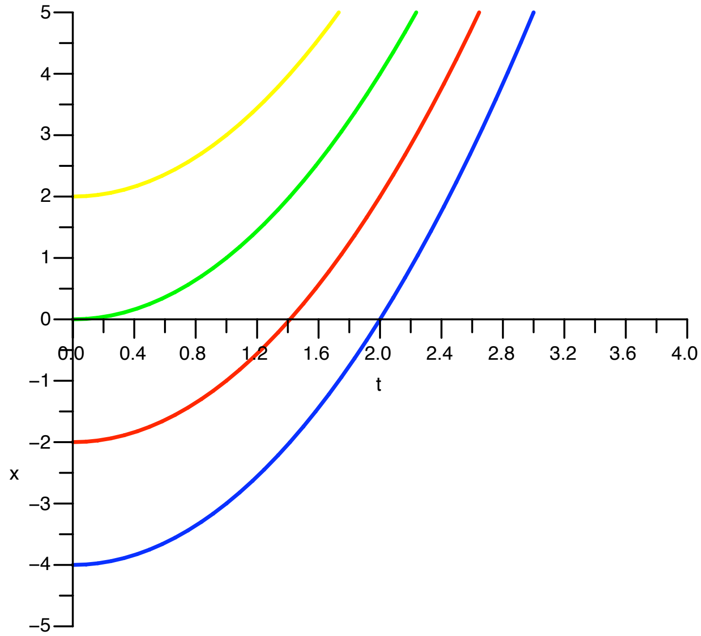
Simple Example: Solve the initial value problem (differential equation plus initial condition)

$$x'(t) = 2t,\ x(0) = 1$$

Step 1: Integrate both sides of the equation to find $x(t)$.
The correct anti-derivative is

1. $x(t) = 2t^2 + C$
2. $x(t) = t^2$
3. $x(t) = t^2 + C$
4. $x(t) = 3t^2$

Correct answer $x(t) = t^2 + C$. This is a family of curves with a different curve for each value of C.



Step 2: Choose the value of C so that $x(0) = 1$.

1. $C = 0$
2. $C = 1$
3. $C = -1$
4. $C = 2$

Learning Objective #1: See how a population can be modeled by an initial value problem and see how to solve it by using the partial fraction integration method.

Simple Growth Model. The simplest model will only concern itself with the effects of births and deaths. Let $x(t)$ be the species population at any time t. The rate of change of $x(t)$ with respect to time is given by

$$\frac{dx}{dt} = \text{birth} \; - \text{deaths}$$

Assume that the birth term is proportional to x which is quite reasonable as we expect the number of births to increase with x. Similarly assume the death term is also proportional to x. Then our differential equation for $x(t)$ is

$$\frac{dx}{dt} = bx - \mu x = (b - \mu)x$$

where b is the per capita birth rate and μ is the per capita death rate. Both these constants have the units of time^{-1}. It is natural to define a net growth rate constant

$$r = b - \mu$$

and then the differential equation for $x(t)$ to be solved is

$$\frac{dx}{dt} = rx.$$

Before we solve this let us notice some qualitative features of this equation. Remember that the population satisfies $x(t) \geq 0$ with $x(t) = 0$ only when the species becomes extinct. Suppose the per capita birth rate is larger than the death rate so that $r > 0$. Then we have

$$\frac{dx}{dt} \geq 0,$$

which means that x is an increasing function of t. This makes sense since there are more births than deaths and so the population will grow.

Question 1: Suppose $r < 0$. What does this mean about births and deaths?

1. birth rate > death rate
2. birth rate < death rate
3. birth rate = death rate

Question 2: In this case is x an increasing or decreasing function of time?

1. increasing
2. decreasing

Solution:

1. Divide both sides by x:
$$\frac{1}{x}\frac{dx}{dt} = r$$

2. Integrate both sides with respect to time t. Note that often, differential equations are solved by integrating.
$$\int \frac{1}{x}\frac{dx}{dt} dt = \int r dt$$
The left hand integral can be simplified by changing the variable of integration to x by using the fact that $dx = \frac{dx}{dt}dt$:
$$\int \frac{1}{x} dx = \int r dt$$
The right hand integration is very simple (remember that r is a constant!). The left hand integration is a very simple example of partial fractions with the integrand already a partial fraction. We get
$$\ln|x(t)| = rt + C$$
where C is the integration constant. Note that the absolute value symbols are not required since $x(t) > 0$. We would like to solve for $x(t)$, so we take the exponential of both sides and get
$$x(t) = e^{rt+C} = e^C e^{rt} = C_0 e^{rt}.$$
Note that e^C is a constant which we will define as C_0, that we don't know yet. To find C_0 we will have give one more piece of information which is called the initial condition. Suppose the population is given at some particular time (perhaps by counting) then C_0 can be found. Suppose $x(0) = 1000$ is the starting population. Then we have
$$x(0) = 1000 = C_0 e^0 = C_0$$
and the solution is $x(t) = 1000e^{rt}$. A differential equation plus initial condition is called an initial value problem.

3. Interpretation. Suppose $r > 0$ (births most important) then $x(t)$ is a growing exponential function with
$$\lim_{t\to\infty} x(t) = \infty.$$
This means a population explosion with no way to stop the growth. But we know that other factors will come into play. We have not included the fact that there are limited resources of food and land into our mathematical model.

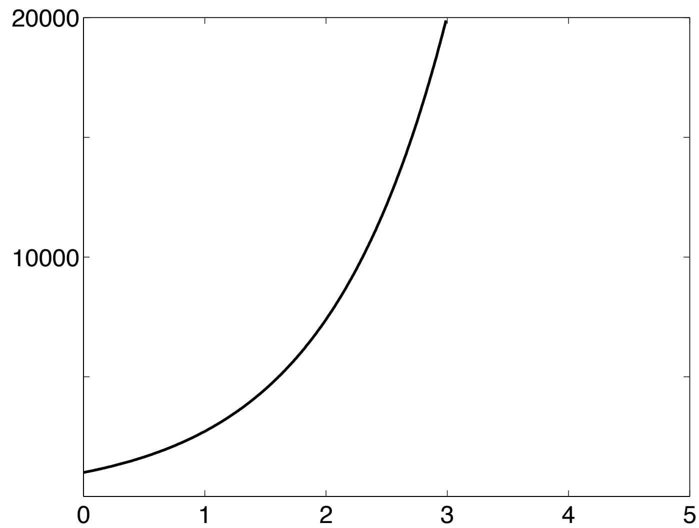


Figure 1: Population as a function of time for $r = 1$.

Question 3: Suppose $r < 0$. What happens to the population in the limit $t \to \infty$?

1. $x(t)$ approaches ∞.
2. $x(t)$ approaches 1000
3. $x(t)$ approaches 0

Question 4: For this case, interpret this result from a biological point of view?

1. Population explosion
2. Population stays relatively constant
3. Population becomes extinct

Lesson Objective #2: Improvement of the mathematical model. Equilibrium solutions. Interpretation of the solution.

The simple model we have been examining can be improved by building into our differential equation a way to prevent the runaway growth when $r > 0$. We want the death rate to increase when x starts to become too large. This is reasonable as we start to use up our resources when the population is too large. Verhulst proposed the following differential equation as a mathematical model:

$$\frac{dx}{dt} = rx(1 - x/K) = rx - rx^2/K.$$

K is called the carrying capacity of the population, which is the largest sustainable population. Note now that when x is large that the second term dominates the first (why?) so the derivative $\frac{dx}{dt}$ is now negative and so the population will decrease.

Before finding a solution to this equation note that $x = 0$ and $x = K$ solve this differential equation. These solutions are called equilibrium or steady state solutions. Equilibrium solutions are found by setting the derivative equal to zero which gives a constant solution. The result is that you solve an algebraic equation.

$$\frac{dx}{dt} = 0 \qquad \Rightarrow \qquad rx(1 - x/K) = 0.$$

Solution by Integration. To simplify our integration take $K = 1$. Then we have

$$\frac{dx}{dt} = rx(1 - x).$$

Note that the equilibrium solutions are $x = 0$ and $x = 1$.

Following the above procedure

1. Divide both sides by $x(1 - x)$:

$$\frac{1}{x(1-x)} \frac{dx}{dt} = r$$

2. Integrate both sides with respect to time t. Note that often, differential equations are solved by integrating.

$$\int \frac{1}{x(1-x)} \frac{dx}{dt} dt = \int r dt$$

As before, the left hand integral can be simplified by changing the variable of integration to x:

$$\int \frac{1}{x(1-x)} dx = \int r dt$$

The left hand integration is done by using partial fractions:

$$\frac{1}{x(1-x)} = \frac{-1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}$$

This implies $-1 = A(x-1) + Bx$ and gives $A = 1$ and $B = -1$. So

$$\int \Big(\frac{1}{x} - \frac{1}{x-1}\Big) dt = \ln|x| - \ln|x-1| = rt + C.$$

Combining the logarithms

$$\ln\left|\frac{x}{x-1}\right| = rt + C$$

and inverting we have

$$\frac{x}{x-1} = e^C e^{rt} = C_0 e^{rt}.$$

Let's suppose the initial population is $x(0) = 2$, then the constant C_0 can be found

$$\frac{2}{2-1} = C_0 e^0 \qquad \Rightarrow \qquad C_0 = 2.$$

and then $x/(x-1) = 2e^{rt}$. Since we don't have $x(t)$ given explicitly we algebraically solve to get

$$x(t) = \frac{2e^{rt}}{2e^{rt}-1} = \frac{2}{2-e^{-rt}}$$

which is the population of our species for any time t.

3. Interpretation. Let's see what happens now when $r > 0$. Evaluating the large time behavior we see that

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} \frac{2}{2-e^{-rt}} = 1.$$

This means a population tends toward the equilibrium population $x(t) = 1$ which is the carrying capacity for our population. This is typical behavior in many biological problems. In fact for any choice of $C_0 \neq 0$, this is true since in general

$$x(t) = \frac{C_0 e^{rt}}{C_0 e^{rt} - 1} = \frac{C_0}{C_0 - e^{-rt}}$$

which has limit 1 as $t \to \infty$ for any starting population $C_0 \neq 0$.

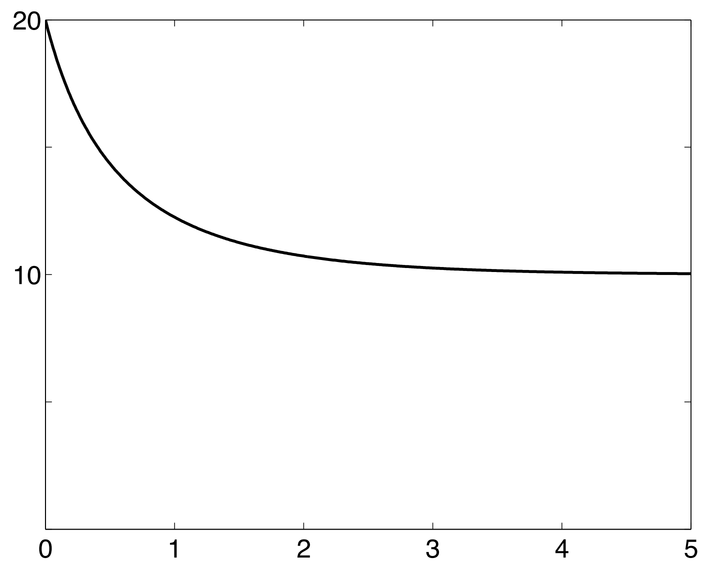


Figure 2: Population as a function of time for $r = 1$.