LAPLACE TRANSFORMS AND INITIAL VALUE PROBLEMS, PART II

Dr. Miao-jung Ou, Department of Mathematics

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PLAN

- Calculus topics: Integration by parts and L'Hospital's Rule
- Definition of Laplace Transforms
- Calculation of Laplace Transforms of basic functions, which include polynomials, sine/cosine functions, exponential functions and their combinations.
- Laplace Transform of derivatives and its application on solving Initial value problems for differential equations
- Auto pilot problems

Example 1. *Calculate the Laplace transform of $f(t) = e^t$.*

$$\begin{aligned}
\mathcal{L}\{e^t\}(s) &:= \lim_{N\to\infty} \int_0^N e^{-st} \cdot e^t dt = \lim_{N\to\infty} \int_0^N e^{(1-s)t} \\
&= \lim_{N\to\infty} \left. \frac{e^{(1-s)t}}{1-s} \right|_0^N \quad \textit{for } s \neq 1 \\
&= \lim_{N\to\infty} \left(\frac{e^{(1-s)N}}{1-s} - \frac{1}{1-s} \right) = \frac{1}{s-1} \quad \textit{for } 1-s < 0, \ \textit{i.e. for } s > 1.
\end{aligned}$$

Example 1. *Calculate the Laplace transform of $f(t) = e^t$.*

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&= \lim_{N\to\infty} \left.\frac{e^{(1-s)t}}{1-s}\right|_0^N \quad \text{for } s \neq 1 \\
&= \lim_{N\to\infty}\left(\frac{e^{(1-s)N}}{1-s} - \frac{1}{1-s}\right) = \frac{1}{s-1} \quad \text{for } 1-s<0, \text{ i.e. for } s>1.
\end{aligned}$$

Question 1. *1 min. 30 sec.* *The Laplace transform of e^{-2t} is*

$(A)\ \frac{1}{s-2},\ s>2,\quad (B)\frac{1}{s+2},\ s>-2$

$(C)\ \frac{-1}{s+2},\ s>-2$ (D) *None of the above.*

$\mathbf{f(t)} = \mathcal{L}^{-1}\{F(s)\}(t)$	$F(s) := \mathcal{L}\{f(t)\}(s)$
t^n, $n = 0, 1, 2, \ldots$	$\frac{n!}{s^{n+1}}$, $s > 0$
$\sin bt$	$\frac{b}{s^2 + b^2}$, $s > 0$
e^{at}	$\frac{1}{s-a}$, $s > a$
$t^n e^{at}$, $n = 1, 2, \ldots$	$\frac{n!}{(s-a)^{n+1}}$, $s > a$
$e^{at} \sin bt$	$\frac{b}{(s-a)^2 + b^2}$, $s > a$
$e^{at} \cos bt$	$\frac{s-a}{(s-a)^2 + b^2}$, $s > a$

Note: $\mathcal{L}^{-1}$ denotes the inverse Laplace transform.

Example 2.

$$\mathcal{L}\left\{e^{3t} + 7e^{2t}\sin 5t - t^3 + 1\right\} = \frac{1}{s-3} + \frac{35}{(s-2)^2 + 25} - \frac{6}{s^4} + \frac{1}{s}$$

$$\text{for } \{s > 3\} \cap \{s > 2\} \cap \{s > 0\} = \{s > 3\}.$$

$\mathbf{f(t)} = \mathcal{L}^{-1}\{F(s)\}(t)$	$F(s) := \mathcal{L}\{f(t)\}(s)$
$t^n e^{at}$, $n = 0, 1, 2, \ldots$	$\dfrac{n!}{(s-a)^{n+1}}$, $s > a$
$e^{at} \sin bt$	$\dfrac{b}{(s-a)^2 + b^2}$, $s > a$
$e^{at} \cos bt$	$\dfrac{s-a}{(s-a)^2 + b^2}$, $s > a$

Question 2. *2 min.*

$$\mathcal{L}\left\{e^{3t} + e^{-2t}\cos 5t + 3t^2 - 5\right\} =$$

(A) $\quad \dfrac{1}{s-3} + \dfrac{s-5}{(s+2)^2+25} + \dfrac{6}{s^3} - \dfrac{5}{s}$, $s > 3$

(B) $\quad \dfrac{1}{s-3} + \dfrac{s+2}{(s+2)^2+25} + \dfrac{6}{s^3} - \dfrac{5}{s}$, $s > 3$

(C) $\quad \dfrac{1}{s+3} + \dfrac{s+2}{(s+2)^2+25} + \dfrac{6}{s^3} - \dfrac{5}{s}$, $s > 3$

(D) $\quad$ *None of the above.*

Inverse Laplace Transform

$\mathbf{f(t)} = \mathcal{L}^{-1}\{F(s)\}(t)$	$F(s) := \mathcal{L}\{f(t)\}(s)$
$t^n e^{at}$, $n = 0, 1, 2, \ldots$	$\dfrac{n!}{(s-a)^{n+1}}$, $s > a$
$e^{at} \sin bt$	$\dfrac{b}{(s-a)^2 + b^2}$, $s > a$
$e^{at} \cos bt$	$\dfrac{s-a}{(s-a)^2 + b^2}$, $s > a$

Example 3.

$$\mathcal{L}^{-1}\left\{\frac{-1}{s+1}\right\} = -\mathcal{L}^{-1}\left\{\frac{1}{s-(-1)}\right\} = -e^{-t}$$

Example 4.

$$\mathcal{L}^{-1}\left\{\frac{2}{(s-1)^2 + 2^2}\right\} = e^t \sin 2t.$$

$f(t) = \mathcal{L}^{-1}\{F(s)\}(t)$	$F(s) := \mathcal{L}\{f(t)\}(s)$
$t^n e^{at}$, $n = 0, 1, 2, \ldots$	$\frac{n!}{(s-a)^{n+1}}$, $s > a$
$e^{at} \sin bt$	$\frac{b}{(s-a)^2 + b^2}$, $s > a$
$e^{at} \cos bt$	$\frac{s-a}{(s-a)^2 + b^2}$, $s > a$

Question 3. *1 min. 30 sec.*

$$\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2 + 3^2}\right\} =$$

(A) $e^t \sin 3t$

(B) $e^t \cos 3t$

(C) $e^{3t} \cos t$

(D) *None of the above.*

Inverse Laplace Transform... Continued

$\mathbf{f(t)} = \mathcal{L}^{-1}\{F(s)\}(t)$	$F(s) := \mathcal{L}\{f(t)\}(s)$
$t^n e^{at}$, $n = 0, 1, 2, \ldots$	$\dfrac{n!}{(s-a)^{n+1}}$, $s > a$
$e^{at} \sin bt$	$\dfrac{b}{(s-a)^2 + b^2}$, $s > a$
$e^{at} \cos bt$	$\dfrac{s-a}{(s-a)^2 + b^2}$, $s > a$

Example 5.

$$\mathcal{L}^{-1}\left\{\frac{-1}{2s+1}\right\} = -\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+\frac{1}{2}}\right\} = -\frac{1}{2}e^{-\frac{1}{2}t}$$

Example 6.

$$\mathcal{L}^{-1}\left\{\frac{s+5}{(s-1)^2+2^2}\right\} = \mathcal{L}^{-1}\left\{\frac{(s-1)+6}{(s-1)^2+2^2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{(s-1)}{(s-1)^2+2^2}\right\} + \mathcal{L}^{-1}\left\{\frac{6}{(s-1)^2+2^2}\right\} = e^t \cos 2t + 3e^t \sin 2t.$$

$\mathrm{f(t)} = \mathcal{L}^{-1}\{F(s)\}(t)$	$F(s) := \mathcal{L}\{f(t)\}(s)$
$t^n e^{at}$, $n = 0, 1, 2, \ldots$	$\dfrac{n!}{(s-a)^{n+1}}$, $s > a$
$e^{at} \sin bt$	$\dfrac{b}{(s-a)^2 + b^2}$, $s > a$
$e^{at} \cos bt$	$\dfrac{s-a}{(s-a)^2 + b^2}$, $s > a$

Question 4. *2 min.*

$$\mathcal{L}^{-1}\left\{\frac{s+1}{(s-1)^2+2^2}\right\} =$$

(A) $e^t \cos 2t + e^t \sin 2t$

(B) $e^t \cos 2t + 2e^t \sin t$

(C) $e^t \cos 2t + \sin 2t$

(D) *None of the above.*

Laplace transform of the derivative of a function

Definition 1. *A function $f(t)$ is said to be of* ***exponential order*** *a if it grows at most as fast as the function e^{at} for large t.*

For example, $f(t) = t^2$ is of order a for any $a > 0$ because $\lim_{t\to\infty} \frac{t^2}{e^{at}} = 0$ by l'Hospital's Rule.

Example 7. *Calculate $\mathcal{L}\{f'\}(s)$ for a continuous function $f = f(t)$ of exponential order a.*

$$
\begin{aligned}
\mathcal{L}\{f'\}(s) &:= \lim_{N\to\infty} \int_0^N e^{-st} \cdot f'(t)dt \\
&\stackrel{\text{IBP}}{=} \lim_{N\to\infty} \left[e^{-st} \cdot f(t)\Big|_0^N - \int_0^N \left(-se^{-st} f(t)dt\right)\right] \\
&= \lim_{N\to\infty} \left(e^{-sN} f(N) - e^0 \cdot f(0) + s\int_0^N e^{-st} f(t)dt \right) \\
&= s\mathcal{L}\{f\}(s) - f(0) \text{ for } s > a.
\end{aligned}
$$

$\mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\}(s) - f(0)$ for $s > a$.

Example 8.

$$\begin{aligned}\mathcal{L}\{f''\}(s) &= s \cdot \mathcal{L}\{f'\}(s) - f'(0) \\ &= s[s\mathcal{L}\{f\}(s) - f(0)] - f'(0) \\ &= s^2\mathcal{L}\{f\}(s) - s \cdot f(0) - f'(0).\end{aligned}$$

Note: $f(0)$ and $f'(0)$ are referred to as initial conditions when s represents the time variable.

$\mathbf{f(t)} = \mathcal{L}^{-1}\{F(s)\}(t)$	$F(s) := \mathcal{L}\{f(t)\}(s)$
e^{at}	$\dfrac{1}{s-a},\ s > a$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$

Example 9. *Use Laplace transform to solve the initial value problem*

$$\text{(DiffEq)}\ y'' - 2y' + 5y = -8e^{-t},\ y(0) = 1,\ y'(0) = 0.$$

Let $Y(s) := \mathcal{L}\{y\}$ *(Laplace transform of the solution). Use the Table to obtain*

$$\mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - s.$$
$$\mathcal{L}\{y'\} = sY(s) - y(0) = sY(s) - 1,$$
$$\mathcal{L}\{-8e^{-t}\} = \frac{-8}{s+1}.$$

Applying Laplace transform to both sides of the linear DiffEq leads to the following ***algebraic equation*** *for* $Y(s)$*:*

$$s^2Y - s - 2(sY - 1) + 5Y = \frac{-8}{s+1},$$
$$\Rightarrow (s^2 - 2s + 5)Y(s) = \frac{-8}{s+1} + s - 2 \Rightarrow Y(s) = \frac{s^2 - s - 10}{(s+1)(s^2 - 2s + 5)}.$$

$\mathbf{f(t)} = \mathcal{L}^{-1}\{F(s)\}(t)$	$F(s) := \mathcal{L}\{f(t)\}(s)$
$t^n e^{at}$, $n = 0, 1, 2, \ldots$	$\dfrac{n!}{(s-a)^{n+1}}$, $s > a$
$e^{at} \sin bt$	$\dfrac{b}{(s-a)^2 + b^2}$, $s > a$
$e^{at} \cos bt$	$\dfrac{s-a}{(s-a)^2 + b^2}$, $s > a$

$$Y(s) = \frac{s^2 - s - 10}{(s+1)(s^2 - 2s + 5)}$$

The solution $y = \mathcal{L}^{-1}\left\{\dfrac{s^2 - s - 10}{(s+1)(s^2 - 2s + 5)}\right\}$.

HOW do we inverse transform this using the Table??

$\mathbf{f(t)} = \mathcal{L}^{-1}\{F(s)\}(t)$	$F(s) := \mathcal{L}\{f(t)\}(s)$
$t^n e^{at}$, $n = 0, 1, 2, \ldots$	$\dfrac{n!}{(s-a)^{n+1}}$, $s > a$
$e^{at} \sin bt$	$\dfrac{b}{(s-a)^2 + b^2}$, $s > a$
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Use partial fraction decomposition

$$\frac{s^2 - s - 10}{(s+1)(s^2 - 2s + 5)} = \frac{A}{s+1} + \frac{Bs + C}{s^2 - 2s + 5}$$

$\Rightarrow A = -1,\ B = 2,\ C = -5.$

$$\begin{aligned} y &= \mathcal{L}^{-1}\left\{\frac{s^2 - s - 10}{(s+1)(s^2 - 2s + 5)}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{-1}{s+1} + \frac{2s - 5}{s^2 - 2s + 5}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{-1}{s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{2s - 5}{s^2 - 2s + 5}\right\} \end{aligned}$$

$\mathbf{f(t)} = \mathcal{L}^{-1}\{F(s)\}(t)$	$F(s) := \mathcal{L}\{f(t)\}(s)$
$t^n e^{at}$, $n = 0, 1, 2, \ldots$	$\dfrac{n!}{(s-a)^{n+1}}$, $s > a$
$e^{at} \sin bt$	$\dfrac{b}{(s-a)^2 + b^2}$, $s > a$
$e^{at} \cos bt$	$\dfrac{s-a}{(s-a)^2 + b^2}$, $s > a$

Need to "complete square" the denominator

$$
\begin{aligned}
y &= \mathcal{L}^{-1}\left\{\frac{-1}{s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{2s-5}{s^2-2s+5}\right\} \\
&= -\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{2s-5}{(s-1)^2+4}\right\} \\
&= -\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{2(s-1)-3}{(s-1)^2+4}\right\} \\
&= -\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{2(s-1)}{(s-1)^2+2^2}\right\} + \mathcal{L}^{-1}\left\{\frac{-3}{(s-1)^2+2^2}\right\}
\end{aligned}
$$

$\mathbf{f(t)} = \mathcal{L}^{-1}\{F(s)\}(t)$	$F(s) := \mathcal{L}\{f(t)\}(s)$
$t^n e^{at}$, $n = 0, 1, 2, \ldots$	$\dfrac{n!}{(s-a)^{n+1}}$, $s > a$
$e^{at}\sin bt$	$\dfrac{b}{(s-a)^2+b^2}$, $s > a$
$e^{at}\cos bt$	$\dfrac{s-a}{(s-a)^2+b^2}$, $s > a$

$$
\begin{aligned}
y &= -\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{2(s-1)}{(s-1)^2+2^2}\right\} + \mathcal{L}^{-1}\left\{\frac{-3}{(s-1)^2+2^2}\right\} \\
&= -\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + 2\mathcal{L}^{-1}\left\{\frac{(s-1)}{(s-1)^2+2^2}\right\} - \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{2}{(s-1)^2+2^2}\right\}
\end{aligned}
$$

$\mathbf{f(t)} = \mathcal{L}^{-1}\{F(s)\}(t)$	$F(s) := \mathcal{L}\{f(t)\}(s)$
$t^n e^{at}$, $n = 0, 1, 2, \ldots$	$\dfrac{n!}{(s-a)^{n+1}}$, $s > a$
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$e^{at} \cos bt$	$\dfrac{s-a}{(s-a)^2 + b^2}$, $s > a$

Question 5. *2 min.*

$$y = -\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + 2\mathcal{L}^{-1}\left\{\frac{(s-1)}{(s-1)^2+2^2}\right\} - \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{2}{(s-1)^2+2^2}\right\} =$$

(A) $-e^{t} + 2e^{-t}\cos 2t - \frac{3}{2}e^{-t}\sin 2t$

(B) $-e^{-t} + 2e^{t}\sin 2t - \frac{3}{2}e^{t}\cos 2t$

(C) $-e^{-t} + 2e^{t}\cos 2t - \frac{3}{2}e^{t}\sin 2t$

(D) *None of the above*

Q: How do we check if $y(t) = -e^{-t} + 2e^{t}\cos 2t - \frac{3}{2}e^{t}\sin 2t$ is indeed the solution to the IVP:

$$\text{(DiffEq) } y'' - 2y' + 5y = -8e^{-t},\ y(0) = 1,\ y'(0) = 0?$$

Servomechanism model for an auto pilot system

Such a system applies a torque to the steering shaft so that an airplane will follow a prescribed direction(angle) $g(t)$. As can be expected, the true direction of the airplane $y(t)$ is not always to be the same as the desired $g(t)$ at all time t. When this happens, the servomechanism will measure the deviation between $y(t)$ and $g(t)$ and feed back to the steering shaft a torque.

Consider a servomechanism with feedback **proportional to the deviation** $e(t) := y(t) - g(t)$ **but with opposite sign**. In this way, the airplane can stay in the desired course. According to Newton's second law of motion, the total torque of a rotating object is related to the moment of Inertia I of the object and the angular acceleration $y''(t)$. Hence the servomechanism following the following rule in correcting the direction of an airplane

$$Iy''(t) = -ke(t),\ I,\ k > 0.$$

Example 10. *Determine the deviation function $e(t)$ for the auto pilot if the steering shaft is initially at rest ($y'(0) = 0$) in the zero direction ($y(0) = 0$) and the desired direction is given by $g(t) = a$ (ie. a fixed drection=a straight line).*

The initial value problem is

$$Iy''(t) = -ke(t),\ y(0) = 0,\ y'(0) = 0.$$

Let $Y(s) = \mathcal{L}\{y(t)\}$ and $E(s) = \mathcal{L}\{e(t)\}$. Laplace transform both sides of the DiffEq to get

$$\mathcal{L}\{Iy''(t)\} = I[s^2Y(s) - sy(0) - y'(0)] = Is^2Y(s) = -kE(s).$$

Since $y(t) = e(t) + g(t) = e(t) + a$, we have $Y(s) = E(s) + \mathcal{L}\{a\} = E(s) + \frac{a}{s}$.

$$Is^2\left(E(s) + \frac{a}{s}\right) = -kE(s).$$

Therefore, $E(s) = \dfrac{-Isa}{Is^2 + k} = \dfrac{-sa}{s^2 + \frac{k}{I}}$. *Inverse Laplace transform* E *to conclude that*

$$e(t) = -a\mathcal{L}^{-1}\left\{\frac{s}{s^2 + \frac{k}{I}}\right\} = -a\cos\left(\sqrt{\frac{k}{I}}t\right).$$

Thank you for your attention!