LAPLACE TRANSFORMS AND INITIAL VALUE PROBLEMS

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Application of Calculus, EXCEL Program, Jan. 24, 2007

PLAN

- Calculus topics: Integration by parts and L'Hospital's Rule
- Definition of Laplace Transforms
- Calculation of Laplace Transforms of basic functions, which include polynomials, sine/cosine functions, exponential functions and their combinations.
- Laplace Transform of derivatives and its application on solving Initial value problems for differential equations
- Auto pilot problems

Integration by parts(IBP): a consequence of **product rule**:

$$\int fg'dx = \int (fg)' - f'gdx = fg - \int f'gdx.$$

Example 1. $\int xe^x dx.$

Solution: Let $f = x$ and $g = e^x$. Note $g' = e^x$.

$$\begin{aligned} \int xe^x dx = \int fg'dx &\overset{\text{IBP}}{=} xe^x - \int (x)'e^x dx \\ &= xe^x - \int e^x dx \\ &= xe^x - e^x + c.\square \end{aligned}$$

Question 1. *1 minute*

$$\int x^2 e^x dx =$$

(A) $x^2e^x - 2xe^x - 2e^x + c$

(B) $x^2e^x - 2xe^x + 2e^x + c$

(C) $x^2e^x - 2xe^x + c$

(D) *None of the above*

Example 2. $\int xe^{-x}dx.$

Solution: Let $f = x$ and $g = \int e^{-x}dx = -e^{-x} + c$ so $g' = e^{-x}$. Let $c = 0$.

$$\begin{aligned}\int xe^{-x}dx = \int fg'dx &\overset{\text{IBP}}{=} x(-e^{-x}) - \int (x)'(-e^{-x})dx \\ &= -xe^{-x} + \int e^{-x}dx \\ &= -xe^{-x} - e^{-x} + c. \square\end{aligned}$$

Question 2. *1 minute 30 seconds*

$$\int x^2 e^{-3x} dx =$$

(A) $-3x^2e^{-3x} - 18xe^{-3x} - 54e^{-3x} + c$

(B) $\frac{1}{3}x^2e^{-3x} + \frac{2}{9}xe^{-3x} - \frac{2}{27}e^{-3x} + c$

(C) $-\frac{1}{3}x^2e^{-3x} - \frac{2}{9}xe^{-3x} - \frac{2}{27}e^{-3x} + c$

(D) *None of the above*

Example 3. $\int \sin x \cdot e^{-x} dx.$

Solution: Let $f = \int \sin x dx = -\cos x + c$ and $g = e^{-x}$ so $f' = \sin x$.
Let $c = 0$.

$$\cdots$$

$$= -\frac{e^{-x}}{2}(\cos x + \sin x) + c. \square$$

Question 3. *1 minute 30 seconds*

$$\int \sin x \cdot e^x dx =$$

(A) $\frac{1}{2}e^x(\sin x - \cos x) + c$

(B) $\frac{1}{2}e^x(\sin x + \cos x) + c$

(C) $-\cos x e^x + c$

(D) *None of the above*

L'Hospital's Rule for $\frac{\infty}{\infty}$

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)} \quad \text{if} \quad \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{\infty}{\infty}.$$

Example 4. *For* $s > 0$, $\lim_{N\to\infty} N \cdot e^{-sN} =?$

Solution:

$$\begin{aligned} & \lim_{N\to\infty}\frac{N}{e^{sN}} \\ = & \lim_{N\to\infty}\frac{N'}{(e^{sN})'} \\ = & \lim_{N\to\infty}\frac{1}{s\cdot e^{sN}} \\ = & \ 0. \end{aligned}$$

Question 4. *1 minute*

For $s > 0$, $\lim_{N \to \infty} N^3 \cdot e^{-sN} =$

(A) ∞

(B) 0

(C) 1

(D) *None of the above*

Question 5. *30 seconds*

$$\textit{For } s > 0, \lim_{N\to\infty} N^{100} \cdot e^{-sN} =$$

(A) ∞

(B) 0

(C) 1

(D) *None of the above*

Laplace Transform

Definition 1. *The one-sided Laplace transform of a function f defined on $[0, \infty)$ is a function of s such that*

$$\mathcal{L}\{f\}(s) := \int_0^\infty e^{-st} f(t)dt := \lim_{N\to\infty} \int_0^N e^{-st} f(t)dt.$$

The domain of $\mathcal{L}\{f\}$ is where the integral exists.

Example 5. *Calculate the Laplace transform of $f(t) = 1$.*

$$\begin{aligned} \mathcal{L}\{1\}(s) &:= \lim_{N\to\infty} \int_0^N 1 \cdot e^{-st} dt \\ &= \lim_{N\to\infty} \left. \frac{e^{-st}}{-s} \right|_{t=0}^{t=N} \\ &= \lim_{N\to\infty} \left(\frac{e^{-sN}}{-s} - \frac{1}{-s} \right) = \frac{1}{s}. \end{aligned}$$

So the Laplace transform of the constant function $f(t) = 1$ is $\frac{1}{s}$ for $s > 0$.

Example 6. *Calculate the Laplace transform of $f(t) = t$.*

$$
\begin{aligned}
\mathcal{L}\{t\}(s) &:= \int_0^\infty e^{-st} \cdot t dt \\
&:= \lim_{N\to\infty} \int_0^N te^{-st} dt \\
&\overset{\text{IBP}}{=} \lim_{N\to\infty} \left[\left. \frac{te^{-st}}{-s} \right|_{t=0}^{t=N} - \int_0^N \frac{e^{-st}}{-s} dt \right] \\
&= \lim_{N\to\infty} \left(\frac{Ne^{-sN}}{-s} - \frac{e^{-sN}}{s^2} + \frac{1}{s^2} \right) \\
&= \frac{1}{s^2} \text{ for } s > 0.
\end{aligned}
$$

Note that L'Hospital Rule was applied to evaluate the first term in the limit.

Question 6. *2 minutes* *The Laplace transform of* $f(t) = t^2$ *is*

(A) $\dfrac{2}{s^3}$, $s > 0$

(B) $\dfrac{1}{s^3}$, $s > 0$

(C) $\dfrac{2}{s^3}$, $s < 0$

(D) *None of the above*

For $n = 0, 1, 2, 3, \cdots,$

$$\mathcal{L}\{t^n\}(s) = \frac{n!}{s^{n+1}},\ s > 0.$$

Example 7. *Calculate the Laplace transform of $f(t) = \sin 3t$.*

$$\mathcal{L}\{\sin 3t\}(s) := \lim_{N\to\infty} \int_0^N e^{-st} \sin 3t dt.$$

By IBP, we have

$$\int_0^N e^{-st} \sin 3t dt = \frac{-s \cdot e^{-sN} \cdot \sin 3N - 3\left(e^{-sN} \cdot \cos 3N - 1\right)}{s^2+9} = \cdots$$

Question 7. *1 minute As $N \to \infty$, the limit of above expression is*

(A) $\frac{-s+3}{s^2+9}$, $s > 0$

(B) $\frac{1}{s^2+9}$, $s > 0$

(C) $\frac{3}{s^2+9}$, $s > 0$

(D) *None of the above*