Probability Theory is the branch of science which deals with random features of our world. Here we study how the definite integral and the fundamental theorem of Calculus is used in Probability Theory, which is instrumental in physics, biology, economics, etc.
Find the derivative of the function \( \int 3x^2x^{u^2} - 1u^2 + 1 \, du \).

**Solution:** Look at a more general situation. Let 

\[
np(1 + z^n \int_{x^3}^x) = \frac{xz}{1 - z^n} \int
\]

Using the chain rule, we obtain

\[
(z)f = (z)I \quad \text{and also} \quad ((x)I) = (x)\partial
\]

Note that 

\[
np(n)f \int_x^z = (z)I
\]

Find the derivative \( (x)\partial \). Denote 

\[
np(n)f \int_{(x)\partial} = (x)\partial
\]

**Problem:** Find the derivative of the function
To finish the problem, note that

\[
\int 3x^2 x u^2 - 1 u^2 + 1 du = \int 3x^0 u^2 - 1 u^2 + 1 du - \int 2x^0 u^2 - 1 u^2 + 1 du.
\]

Then, applying the formula

\[
d\left( \int g(x) f(u) du \right) = f(g(x)) g'(x) du
\]

obtain

\[
(x)^g((x)^b) f = np(n) f (x)^b \int \frac{xp}{p} dx
\]

and

\[
np \frac{I + zn}{I - zn} \int xz \frac{xp}{p} - np \frac{I + zn}{I - zn} \int xz \frac{xp}{p} = np \frac{I + zn}{I - zn} xz \int \frac{xp}{p}
\]
Let us start from something familiar. How does this relate to probability theory?

For the discrete random variable $X$, one can define its distribution:

<table>
<thead>
<tr>
<th>Value of $X$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>...</th>
<th>$x_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability</td>
<td>$p_1$</td>
<td>$p_2$</td>
<td>...</td>
<td>$p_n$</td>
</tr>
</tbody>
</table>

Let $A > X > B$. A random variable is called **discrete** if it takes finite number of values. Let $n$.

A random variable is called **discrete** if it takes finite values.

How does this relate to probability theory?
CUMULATIVE DISTRIBUTION FUNCTION (CDF)

On one can calculate any probability associated with $X$, $A < X < B$.
For example, $A < X < B$.

One can calculate any probability associated with $X$, $A < X < B$.

\[ (p)_H - (q)_H = (q \geq X > p)_H \]

The cumulative distribution function (CDF) of $X$ is

\[ F_X(x) = P(X \leq x) = p_1 + p_2 + \cdots + p_l, x_l \leq x \]

where $l \geq 1$ and $i = 1 = (B) X_H \geq 0$, $0 = (A) X_H$.

Observe that $F_X(A) = 0$, $F_X(B) = 1$ and $0 \leq F_X(x) \leq 1$.

The CDF $F_X(x)$ of $X$ is

\[ x \geq l \cdot d + \cdots + 2 \cdot d + 1 \cdot d = (x \geq X)_H = (x) X_H \]

\[ F_X(x) = P(x \geq X)_H = (x) X_H \]

\[ (q \geq X > p)_H \]

The CDF $F_X(x)$ of $X$ is

\[ s + x \cdot d + \cdots + 1 + x \cdot d + x \cdot d = (q \geq X > p)_H \]

\[ (A) X_H \geq 0 \]

For example, $A < X < B$.

\[ (p)_H - (q)_H = (q \geq X > p)_H \]

\[ (p)_H - (q)_H = (q \geq X > p)_H \]
Let $Y = g(X)$ and $y_i = g(x_i)$.

<table>
<thead>
<tr>
<th>$u_d$</th>
<th>$u_f$</th>
<th>...</th>
<th>$u_d$</th>
<th>$u_f$</th>
<th>...</th>
<th>$u_d$</th>
<th>$u_f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1-d$</td>
<td>$1-f$</td>
<td>...</td>
<td>$1-d$</td>
<td>$1-f$</td>
<td>...</td>
<td>$1-d$</td>
<td>$1-f$</td>
</tr>
</tbody>
</table>

Distribution of $X$:

- Probabilities:
  - $P(c < Y \leq d) = p_1 + p_{l+1} + \cdots + p_{l+r}$, where the sum is taken over $i$ such that $c < y_i = g(x_i) < d$.

The cdf of $Y$ is $F_Y(y) = P(Y \leq y) = P(g(X) \leq y) = p_1 + p_{l+1} + \cdots + p_{l+r}$, where the sum is taken over $i$ such that $y_i = g(x_i) \leq y$.

Transformations:
The mean of the random variable $X$ is

$$E(X) = \mu_1 p_1 + \mu_2 p_2 + \cdots + \mu_n p_n.$$
Consider a discrete random variable $X$. The distribution of $X$ is given in the table below.

<table>
<thead>
<tr>
<th>Value of $X$</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.1</td>
</tr>
<tr>
<td>2</td>
<td>0.3</td>
</tr>
<tr>
<td>3</td>
<td>0.05</td>
</tr>
<tr>
<td>4</td>
<td>0.25</td>
</tr>
<tr>
<td>6</td>
<td>0.3</td>
</tr>
<tr>
<td>8</td>
<td>0.1</td>
</tr>
</tbody>
</table>

Also, $EX = 1 \times 0.1 + 2 \times 0.3 + 4 \times 0.05 + 6 \times 0.25 + 8 \times 0.3 = 4.8$.

Also, $P(2 \leq X < 6) = X_F(6) - X_F(2) = 0.7 - 0.4 = 0.3$.

Find $P(2 \leq X < 6)$ using the cdf $X_F(x)$.

Finally, $P(2 \leq X < 6) = X_F(6) - X_F(2)$.

**Example**
Let $Y = X^2$. The distribution of $X$ has the form:

$$
1^2 \cdot 0.1 + 2^2 \cdot 0.3 + 3^2 \cdot 0.3 + 4^2 \cdot 0.25 + 5^2 \cdot 0.05 + 6^2 \cdot 0.04 = 30.3
$$

**Also,**

$$
FY = 1 \cdot 0.1 + 4 \cdot 0.3 + 16 \cdot 0.05 + 36 \cdot 0.25 + 64 \cdot 0.04 = 30.3
$$

<table>
<thead>
<tr>
<th>Value of $Y$</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.05</td>
</tr>
<tr>
<td>4</td>
<td>0.25</td>
</tr>
<tr>
<td>9</td>
<td>0.3</td>
</tr>
<tr>
<td>16</td>
<td>0.1</td>
</tr>
<tr>
<td>36</td>
<td>0.3</td>
</tr>
<tr>
<td>64</td>
<td>0.4</td>
</tr>
</tbody>
</table>

Note that

$$
F_Y(4) = 0.4, \quad F_Y(36) = 0.04, \quad F_Y(6) = 0.03
$$

Also,

$$
\mathbb{E}(Y) = 1 \cdot 0.1 + 2^2 \cdot 0.3 + 3^2 \cdot 0.3 + 4 \cdot 0.25 + 6 \cdot 0.05 + 8 \cdot 0.03 = 30.3
$$

**Example: Continuation**
Airlines frequently overbook flights. Suppose, for a plane with 100 seats, an airline takes 110 reservations. From past experiences, we know the distribution of the number of people with reservations who show up to board the flight. Airlines frequently overbook the flights.

**Airline Example**
Suppose that the airline charges $200 per ticket. So, it is profitable to sell more tickets. Every extra passenger who shows up for the flight but does not have a seat costs the airline $250. $Y$ is the airline profit from overbooking.

$Y = \begin{cases} 10 \times 200 - (X - 100) \times 250 & X > 100 \\ 0 & X \leq 100 \end{cases}$

Here $I(X > 100)$ is the indicator function. $I(X > 100) = 1$ if $X > 100$, $I(X > 100) = 0$ if $X \leq 100$.

$X$ is the number of people with reservations who show up. 

$Y$ is the airline profit from overbooking.
Construct the probability distribution of $Y$.

(use Table 5 on page 7–5).

Calculate the average profit.

GROUP PROBLEM SOLVING (10 MINUTES)
Here \( 0.05 + 0.10 + 0.12 + 0.14 + 0.24 + 0.17 = 0.82 \)

<table>
<thead>
<tr>
<th>( p(y_i) )</th>
<th>( x(\bar{y}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0013</td>
<td>0.0037</td>
</tr>
<tr>
<td>0.005</td>
<td>0.005</td>
</tr>
<tr>
<td>0.005</td>
<td>0.005</td>
</tr>
<tr>
<td>0.01</td>
<td>0.02</td>
</tr>
<tr>
<td>0.02</td>
<td>0.03</td>
</tr>
<tr>
<td>0.03</td>
<td>0.05</td>
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<tr>
<td>0.05</td>
<td>0.06</td>
</tr>
<tr>
<td>0.06</td>
<td>0.07</td>
</tr>
<tr>
<td>0.17</td>
<td>0.14</td>
</tr>
<tr>
<td>0.24</td>
<td>0.24</td>
</tr>
<tr>
<td>0.12</td>
<td>0.12</td>
</tr>
<tr>
<td>0.10</td>
<td>0.10</td>
</tr>
<tr>
<td>0.05</td>
<td>0.05</td>
</tr>
<tr>
<td>0.04</td>
<td>0.04</td>
</tr>
<tr>
<td>0.06</td>
<td>0.07</td>
</tr>
<tr>
<td>0.17</td>
<td>0.17</td>
</tr>
<tr>
<td>0.24</td>
<td>0.24</td>
</tr>
<tr>
<td>0.12</td>
<td>0.12</td>
</tr>
<tr>
<td>0.10</td>
<td>0.10</td>
</tr>
<tr>
<td>0.05</td>
<td>0.05</td>
</tr>
<tr>
<td>0.04</td>
<td>0.04</td>
</tr>
<tr>
<td>I</td>
<td>I</td>
</tr>
</tbody>
</table>

Distribution of the amount of the profit.
Hence, the table can be rewritten like this:

<table>
<thead>
<tr>
<th>y_i</th>
<th>p(y_i)</th>
<th>i</th>
</tr>
</thead>
<tbody>
<tr>
<td>2000</td>
<td>0.82</td>
<td>7</td>
</tr>
<tr>
<td>1750</td>
<td>0.06</td>
<td>7</td>
</tr>
<tr>
<td>1500</td>
<td>0.03</td>
<td>7</td>
</tr>
<tr>
<td>1250</td>
<td>0.02</td>
<td>7</td>
</tr>
<tr>
<td>1000</td>
<td>0.01</td>
<td>7</td>
</tr>
<tr>
<td>750</td>
<td>0.005</td>
<td>8</td>
</tr>
<tr>
<td>500</td>
<td>0.005</td>
<td>9</td>
</tr>
<tr>
<td>-250</td>
<td>0</td>
<td>10</td>
</tr>
<tr>
<td>-500</td>
<td>0</td>
<td>11</td>
</tr>
</tbody>
</table>

Distribution of the amount of profit
The pdf of $Y$ is $F_Y(y) = P(Y \leq y) = P(g(x) \leq y) = \sum_{i} g(x_i)p_i$.

The average profit is $E_Y = \sum_{i} g(x_i)p_i = 2000 \ast 0.82 + 1750 \ast 0.06 + 1500 \ast 0.04 + 1250 \ast 0.03 + 1000 \ast 0.02 + 750 \ast 0.01 + 500 \ast 0.005 + 250 \ast 0.005 + 0 \ast 0.005 - 250 \ast 0.0037 - 500 \ast 0.0013 + 0 \ast 0.005 - 250 \ast 0.005 + 1000 \ast 0.02 + 250 \ast 0.005$.

The pdf of $Y$ is $f_Y(x) = f_X(x) \theta(x)$.

$\int_{0}^{\infty} f_Y(x) \theta(x) \, dx = f(X) \theta(x) d = f(X) \Lambda f$.
What is the probability that the airline does not have any extra expenses due to overbooking? How can this be expressed via the value of the random variable $X$?

What is the probability that overbooking is profitable? How can this be expressed via the value of the random variable $Y$?
Would overbooking be still profitable if the ticket cost $150? Explain.
The average profit is 
Again $0.05 + 0.10 + 0.12 + 0.14 + 0.17 = 0.82.$

<table>
<thead>
<tr>
<th>$y_i$</th>
<th>750</th>
<th>-750</th>
<th>0</th>
<th>250</th>
<th>500</th>
<th>750</th>
<th>1000</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p(y_i)$</td>
<td>0.03</td>
<td>0.02</td>
<td>0.01</td>
<td>0.005</td>
<td>0.005</td>
<td>0.005</td>
<td>0.005</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$x_i$</th>
<th>8</th>
<th>7</th>
<th>6</th>
<th>5</th>
<th>4</th>
<th>3</th>
<th>2</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p(x_i)$</td>
<td>0.04</td>
<td>0.06</td>
<td>0.17</td>
<td>0.24</td>
<td>0.14</td>
<td>0.12</td>
<td>0.10</td>
<td>0.05</td>
</tr>
</tbody>
</table>

Distribution of the amount of the profit

SOLUTION TO FIVE MINUTE PAPER
A random variable \( X \) which may take any value in the finite interval \((A, B)\) is called *continuous*.

How can one calculate the probabilities?

Calculate the cumulative distribution function (cdf) of \( X \).

\[
P(x) = P(x = X) = 0 \quad \text{for any} \quad x \quad \text{is zero.}
\]

\( \)
The cumulative distribution function (CDF) is the probability that $X \leq x$.

One can also evaluate the "instantaneous" probability that $(x) \overset{\text{pdf}}{\Pr} \overset{\text{pdf}}{\Pr} = \overset{\text{pdf}}{\Pr}$ as $x = X$.

The CDF at $X$ is $x$ at $X$. The cumulative distribution function (CDF)
The mean of $X$ can be evaluated as:

$$xp(x)Xf x \int_A^B = X\mathbb{E}$$

$$\int_A^B f_X(x) dx \geq 0$$ since $F_X(x)$ is increasing

$$= \mathbb{E}$$

$$= \mathbb{E}$$

$$= \mathbb{E}$$

The properties of pdf and the mean.
EXAMPLE

Let the p.d.f.

\[ f_X(x) = \frac{3}{2} x + 1 \]

for \( 0 < x < 1 \).

Then the c.d.f.

\[ F_X(x) = \int_0^x f_X(z) \, dz \]

for \( 0 < x < 1 \).

The expectation of \( X \) can be obtained as

\[ \mathbb{E}[X] = \frac{5}{9} \]

and

\[ \mathbb{E}[X] = \frac{4}{3} \text{ for } 0 < x < \frac{1}{2} \]

and

\[ \mathbb{E}[X] = \frac{5}{12} \text{ for } \frac{1}{2} < x < 1 \]

Using the expression for \( F_X(x) \), one can find

\[ P(1/4 < X < 1/2) = F_X(1/2) - F_X(1/4) = \frac{5}{12} - \frac{3}{16} \]

The expectation of \( X \) can be calculated as

\[ \mathbb{E}[X] = \int_0^1 x f_X(x) \, dx \]

for \( 0 < x < 1 \).
The amount of time a student at UCF spends working on Calculus at home every week has the pdf

\[ f_T(t) = \begin{cases} 12t - t^2, & 0 \leq t \leq 12 \\ 0, & \text{otherwise} \end{cases} \]

Using the expression for \( F_T(t) \), find the probability that a student spends less than 4 hours studying Calculus.

Using the expression for \( F_T(t) \), find the cdf \( F_T(t) \) of \( T \) using formula

\[
F_T(t) = \int_0^t f_T(z) \, dz
\]

Evaluate the integral for different values of \( t \). For example, when \( t = 4 \),

\[
F_T(4) = \int_0^4 (12z - z^2) \, dz = \left[ 6z^2 - \frac{z^3}{3} \right]_0^4 = 288 - \frac{64}{3} = \frac{896}{3}
\]

The amount of time \( T \) a student at UCF spends working on Calculus at home every week has the pdf

\( f_T(t) = \begin{cases} 12t - t^2, & 0 \leq t \leq 12 \\ 0, & \text{otherwise} \end{cases} \)
\[
\frac{864}{184^2 - 288^2} = \int_0^1 \frac{888}{3^z} - \frac{288}{z^2} \, dz = \int_0^1 z \cdot p(z) \, dz
\]

\[
F(t) = 0.2593 = (t > 4)
\]

\[
P(t) = (t < 4)
\]

\textbf{Answers}
Find the expectation of $T$ using formula

$$\mathbb{E}_T = \int_0^1 f_T(t) dt.$$ 

What does this expectation show?
The fact that $ET = 6$ means that on the average students spend 6 hours a week on Calculus.

$$ET = 6 \Rightarrow \int_{0}^{12} tf(T) dt = \int_{0}^{12} 12t^2 - t^3 dt$$

$$= \left[ 4t^3 - \frac{1}{4}t^4 \right]_{0}^{12} \frac{288}{1} = 6.$$
Let $X$ have the pdf $f_X(x)$ and the cdf $F_X(x)$.

Objective: Evaluate the cdf $F_Y(y)$ and the pdf $f_Y(y)$.

In both cases, $(x)_b$ has an inverse function $x$ of $(y)_b$ be a monotone (increasing or decreasing) function of $x$.

The variable of interest is $Y = g(X)$.

Let $X$ have the pdf $f_X(x)$ and the cdf $F_X(x)$.
Let $g(x)$ be an increasing function of $x$.

Hence,\[ (h)_y((h)_y)Xf = (h)_y \]

To find the pdf of $Y$, take the derivative of both sides\[ ((h)_y)_X = ((h)_y \triangleright X) d = ((h)_y \triangleright ((X)_b)_y) d = \]
\[ (h \triangleright (X)_b)_y d = (h \triangleright \lambda)_y d = (h)_y \]

Then $h(y)$ is also an increasing function and

**Evaluation of CDF and PDF**
Now, let $g(x)$ be a decreasing function. Then $(h(y))^Y \geq h(y)$ is also decreasing and $(h(y))^X \leq h(y)$. Then

\[
(h)_Y (((h)_Y)^X f) - \equiv (h)_Y f
\]

Taking the derivative of both sides, obtain:

\[
((h)_Y)^X \mathcal{F} - 1 = ((h)_Y > X) d - 1 =
((h)_Y \leq X) d = ((h)_Y \leq ((X)_b)^Y) d =
(h \geq (X)_b) d = (h \geq \lambda) d = (h)_Y \mathcal{P}
\]
We have

\[ f_Y(y) = \begin{cases} 
  f_X(h(y)) & \text{if } h(y) \text{ is increasing} \\
  -f_X(h(y)) & \text{if } h(y) \text{ is decreasing} \\
  |h'(y)|f_X(h(y)) & \text{otherwise}
\end{cases} \]

where \( C > \lambda > D \).

The expectation of \( \lambda \) can be evaluated as

\[ \lambda \mathbb{E} = \int_D^C h_Y(\lambda) f_Y(y) dy \]

We have

**The PDF AND THE MEAN**
THE MEAN (CONTINUATION)

The expectation of $Y$ can also be found as

$$EY = \int_A^B g(x)f_X(x)dx.$$  

Show that the expressions are equal

Consider the case when $g(x)$ is increasing.
Then $C = g(A)$, $D = g(B)$.

Use substitution: $h(y) = x$, $h'(y)dy = dx$, $y = g(x)$, $h(g(A)) = A$, $h(g(B)) = B$.

$$EY = \int_C^D yf_Y(y)dy = \int_{g(A)}^{g(B)} yf_X(h(y))h'(y)dy$$
$$= \int_A^B g(x)f_X(x)dx.$$
Consider variable \( X \) with the pdf

\[
f_X(x) = \frac{2(x+1)}{3}, \quad 0 < x < 1.
\]

Let \( Y = X^2 \), i.e., \( g(x) = x^2 \) and \( h(y) = \sqrt{y}, 0 < y < 1 \).

Recall that

\[
(x) X_H = \frac{\frac{\alpha}{1} \alpha + 1}{\frac{\alpha}{1} \alpha + 1} = (\alpha)^{1 \alpha}(1 + x) \int_0^\frac{\alpha}{1} \frac{\alpha}{1} x \, dx = (x) X_f
\]

\( \forall \, \alpha > 1, \, \alpha > 0, \, \alpha \leq \alpha \), i.e., \( X \Rightarrow Y \)

\[
\frac{\frac{3}{1}}{\frac{3}{1} + 1} = (\alpha)^{\frac{\alpha}{1}}(1 + x) \int_0^\frac{\alpha}{1} \frac{\alpha}{1} x \, dx = (x) X_f
\]

The pdf of \( Y \) calculated by taking the derivative of \( F_Y(y) \):

\[
f_Y(y) = \frac{f_X(x)}{2x} \Big|_{x=\sqrt{y}} = \frac{2\sqrt{y}(\sqrt{y}+1)}{3y}, \quad 0 < y < 1.
\]
The pdf of 

\[ f_Y(y) = \int_1^\infty \frac{f_X(z)}{z} \, dz = \frac{x^2}{(1 + x)^2} \int_1^\infty \frac{z^2}{z} \, dz = \frac{1}{2} \left( \frac{1}{y} - \frac{1}{y^2} \right) \]

The expectation of \( Y \) can be found as

\[ \mathbb{E}Y = \int_0^\infty \frac{x}{1 + x^2} \, dx = \frac{1}{3} \int_1^\infty \frac{1}{z} \, dz = \frac{1}{2} \left( 1 + \ln 2 \right) \approx 0.500 \]

or

\[ \mathbb{E}Y = \int_0^\infty \frac{x}{1 + x^2} \, dx = \frac{1}{3} \int_1^\infty \frac{1}{z} \, dz = \frac{1}{2} \left( 1 + \ln 2 \right) \approx 0.500 \]

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or

\[ \mathbb{E}Y = \int_1^\infty \frac{x}{1 + x^2} \, dx = \frac{1}{3} \int_1^\infty \frac{1}{z} \, dz = \frac{1}{2} \left( 1 + \ln 2 \right) \approx 0.500 \]

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\[ \mathbb{E}Y = \int_1^\infty \frac{x}{1 + x^2} \, dx = \frac{1}{3} \int_1^\infty \frac{1}{z} \, dz = \frac{1}{2} \left( 1 + \ln 2 \right) \approx 0.500 \]

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Recall that the amount of time $T$ a student at UCF spends working on Calculus at home every week has the pdf

$$f_T(t) = \begin{cases} t^2 - 12t + 288, & 0 \leq t \leq 12 \\ 0, & \text{otherwise} \end{cases}$$

The percentage of the final grade $Y$, $0 \leq Y \leq 100$, is related to the time a student spends working on Calculus at home as

$$Y = 100 \sqrt{\frac{t}{12}}.$$ 

As the amount of time $T$ a student at UCF spends working on Calculus at home every week has the pdf
1. Using the expression for $F_T(t)$, find the probability that a student works less than 6 hours a week, works less than 3 hours a week, works more than 9 hours a week.

2. Find the inverse $t = h(y)$ of $y = g(t)$.

3. Find the cdf $F_Y(y)$ of the final percentage $Y$ using the reasoning we employed in the general case. Write the explicit expression for $F_Y(y)$ as a final answer.

4. Using $F_Y(y)$, find the probability that a student has less than 70% (passes the course), more than 90% (gets an A).
5. Find the p.d.f $f_Y(y)$ of the final percentage $Y$ by taking the derivative of $F_Y(y)$. Compare the result of differentiation with $f_Y(y)$ found using formula $f_Y(y) = -f_X(h(y))h'(y)$.

6. Graph the p.d.f $f_T(t)$ and $f_Y(y)$ on separate graphs. Is $f_T(t)$ symmetric? Is $f_Y(y)$ symmetric? What does this mean?

7. Find the expectation of $Y$ using formulae $E(Y) = \int D_C y f_Y(y) \, dy$ and $E(Y) = \int B_A g(x) f_X(x) \, dx$. Check that both formulae give the same result.

8. How much work does the student need to put in order to get at least a "B" in the course?